

THE CODING OF COMPACT REAL TREES BY REAL VALUED FUNCTIONS

by
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Abstract

This paper is a detailed study of the coding of real trees by real valued functions that is motivated by probabilistic problems related to continuum random trees. Indeed it is known since the works of Aldous [5] and Le Gall [33] that a continuous non-negative function h on $[0, 1]$ such that $h(0) = 0$ can be seen as the contour process of a compact real tree. This particular coding of a compact real tree provides additional structures, namely a root that is the vertex corresponding to $0 \in [0, 1]$, a linear order inherited from the usual order on $[0, 1]$ and a measure induced by the Lebesgue measure on $[0, 1]$; of course, the root, the linear order and the measure obtained by such a coding have to satisfy some compatibility conditions. In this paper, we prove that any compact real tree equipped with a root, a linear order and a measure that are compatible can be encoded by a non-negative function h defined on a finite interval $[0, M]$, that is assumed to be left-continuous with right-limit, without positive jump and such that $h(0+) = h(0) = 0$. Moreover, this function is unique if we assume that the exploration of the tree induced by such a coding backtracks as less as possible. We also prove that a measure-change on the tree corresponds to a re-parametrization of the coding function. In addition, we describe several path-properties of the coding function in terms of the metric properties of the real tree.

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1 Introduction

Real trees form a class of loop-free length spaces, which turns out to be the class of limiting objects of many combinatorial and discrete trees. More precisely, we say that a metric space (T, d) is a *real tree* if it satisfies the following conditions:

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- For all $\sigma, \sigma' \in T$, there is an isometry $f_{\sigma, \sigma'} : [0, d(\sigma, \sigma')] \rightarrow T$ such that $f_{\sigma, \sigma'}(0) = \sigma$ and $f_{\sigma, \sigma'}(d(\sigma, \sigma')) = \sigma'$. We introduce the following notation

$$\llbracket \sigma, \sigma' \rrbracket := f_{\sigma, \sigma'}([0, d(\sigma, \sigma')]) .$$

- If q is a continuous injective map from $[0, 1]$ into T , we have

$$q([0, 1]) = \llbracket q(0), q(1) \rrbracket .$$

Let us introduce some notation: we denote by $\llbracket \sigma, \sigma' \rrbracket$, $\llbracket \sigma, \sigma' \rrbracket$ and $\llbracket \sigma, \sigma' \rrbracket$ the images of resp. $(0, d(\sigma, \sigma'))$, $[0, d(\sigma, \sigma'))$ and $(0, d(\sigma, \sigma'))$ by $f_{\sigma, \sigma'}$. For any $\sigma \in T$ we denote by $n(\sigma, T)$ the *degree* of σ , namely the (possibly infinite) number of connected components of $T \setminus \{\sigma\}$. For convenience of notation, we often denote $n(\sigma, T)$ simply by $n(\sigma)$ when there is no risk of confusion. We denote by

$$\text{Lf}(T) = \{\sigma \in T \setminus \{\rho\} : n(\sigma, T) = 1\} \quad \text{and} \quad \text{Br}(T) = \{\sigma \in T \setminus \{\rho\} : n(\sigma, T) \geq 3\}$$

respectively the set of *leaves* of T and the set of *branching points* of T . By convention, the root ρ is neither a leaf nor a branching point. We also denote by $\text{Sk}(T)$ the *internal skeleton* of T : $\text{Sk}(T) = T \setminus \text{Lf}(T)$. We can easily prove that for any sequence σ_n , $n \geq 1$, that is dense in T , we have

$$\text{Sk}(T) = \bigcup_{n \geq 1} \llbracket \rho, \sigma_n \rrbracket . \quad (1)$$

Since T is compact, we easily show that $\text{Br}(T)$ is at most countable (see Lemma 3.1 in [20]). We shall also need to introduce *the length measure* of a real tree (T, d) denoted by ℓ_T . The length measure is defined on the trace on $\text{Sk}(T)$ of the Borel sigma-field of T and it is characterized by

$$\ell_T(\llbracket \sigma, \sigma' \rrbracket) = d(\sigma, \sigma') .$$

The length measure can also be seen as the one-dimensional Hausdorff measure on T .

Real trees have a characterization called the *four points condition* that asserts that if (X, d) is complete path-connected metric space then it is a real tree iff

$$d(\sigma_1, \sigma_2) + d(\sigma_3, \sigma_4) \leq (d(\sigma_1, \sigma_3) + d(\sigma_2, \sigma_4)) \vee (d(\sigma_1, \sigma_4) + d(\sigma_2, \sigma_3)), \quad (2)$$

for all $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in T$. The four points condition has been first investigated independently by K. A. Zareckii [45], J.M.S. Simões Pereira [43] and P. Buneman [8]. See A. Dress, V. Moulton and W. Terhalle [12, 13, 14] and also [10, 38] for general results concerning real trees. We also refer to the works of D. Aldous [4, 5] and of J-F. Le Gall [34] for a first study of the Brownian Continuum Random Tree (CRT for short) that is coded by the normalized Brownian excursion. We refer to S. N. Evans [21] for the first explicit use of real tree to construct random trees (see also [22, 23]); the reader interested in applications to phylogenetic models may consult the books of J. Felsenstein [24] and of C. Semple and M. Steel [42]; in a different direction, for applications to Super-Brownian motion, see [33, 32]; see also [29, 11] for a study of the Brownian motion on the Brownian CRT. We refer to the work of J-F. Le Gall and Y. Le Jan [35] for the definition of Lévy trees that are random real trees generalizing Aldous's Brownian CRT; see also [36, 17]

for application to general superprocesses and [15, 18, 19, 20, 44] for fractal and probabilistic properties of Lévy trees. We refer to the work of D. Aldous, J. Pitman and G. Miermont [1, 2, 3] for a detailed account on inhomogeneous continuum random trees (inhomogeneous continuum random trees generalize the CRT and they are the possible scaling limits of interesting discrete combinatorial trees in connection with random mappings). See the papers of B. Hass and G. Miermont [39, 40, 25] for fragmentation processes linked with real trees. Let us also mention that in [26] T. Lyons and B. Hambly use real trees and tree-like paths for rough path integration theory.

It has been shown by S.N. Evans, J. Pitman and A. Winter in [22] that the set of isometry classes of compact real trees endowed with the Gromov-Hausdorff distance is a complete separable metric space. However there seems to be no natural way to choose a representant in a given isometry class. This contrasts with the discrete case. Indeed, if we consider a finite ordered rooted tree that is a finite planar graph without cycle with a distinguished vertex, then it is possible to label its vertices with words written with positive integers (see [41]). More precisely, set $\mathbb{U} = \{\emptyset\} \cup \bigcup_{n \geq 1} (\mathbb{N}^*)^n$, where \mathbb{N}^* is the set $\{1, 2, \dots\}$ of positive integers and where \emptyset stands for the empty word; an ordered rooted tree can be viewed as a subset τ of \mathbb{U} satisfying the following conditions:

- (i) $\emptyset \in \tau$ and \emptyset is called the *root* of τ .
- (ii) If $v = (v_1, \dots, v_n) \in \tau$ then, $(v_1, \dots, v_k) \in \tau$ for any $1 \leq k \leq n$.
- (iii) For every $v = (v_1, \dots, v_n) \in \tau$, there exists $k_v(\tau) \geq 0$ such that $(v_1, \dots, v_n, j) \in \tau$ for every $1 \leq j \leq k_v(\tau)$.

If $v = (v_1, \dots, v_n) \in \tau$, then $|v| = n$ is its *height* in τ , that is its distance from the root (so we set $|\emptyset| = 0$). Observe that \mathbb{U} is linearly (or totally) ordered by the *lexicographical order* denoted by \leq . If τ is finite, then we can list its vertices in an increasing sequence with respect to the lexicographical order, namely $\emptyset = v(0) < v(1) < \dots < v(\#\tau - 1)$. We define the *height process* of τ by

$$H_n(\tau) = |v(n)|, \quad 0 \leq n < \#\tau.$$

Clearly $H(\tau)$ characterizes the tree τ and in particular for any $0 \leq m \leq n < \#\tau$, the youngest common ancestor of $v(m)$ and $v(n)$ is situated at height $\min\{H_k(\tau); m \leq k \leq n\}$. Thus, the distance between $v(m)$ and $v(n)$ is given in terms of $H(\tau)$ by

$$\text{dist}(v(m), v(n)) = H_m(\tau) + H_n(\tau) - 2 \min_{m \leq k \leq n} H_k(\tau).$$

One of the aim of this paper is to provide a similar coding for compact real trees and also an uniqueness result for such a representation. It turns out that the relevant class of coding functions for compact real trees are the left-continuous with right-limit functions: such functions are called *caglad functions* in the standard probabilistic terminology (caglad standing for “continu à gauche et avec limite à droite” in french). We shall explain further why the set of caglad functions is the right class of coding functions to consider (see Comment 1.1).

Let us be more specific: for any $M \geq 0$, let us denote by \mathcal{H}_M the set of non-negative caglad functions h on $[0, M]$ such that $h(0) = h(0+) = 0$ and $h(t) - h(t+) \geq 0$, $t \in [0, M)$. The set

$\mathcal{H} = \cup_{M \geq 0} \mathcal{H}_M$ is called the *set of height functions* and if $h \in \mathcal{H}_M$, $\zeta(h) = M$ is called the *lifetime* of h . Let $h \in \mathcal{H}_M$. For every $s, t \in [0, M]$, we set

$$m_h(s, t) = \inf_{r \in [s \wedge t, s \vee t]} h(r)$$

and

$$d_h(s, t) = h(s) + h(t) - 2m_h(s, t).$$

We can easily show that for any $s_1, s_2, s_3, s_4 \in [0, M]$ we get

$$d_h(s_1, s_2) + d_h(s_3, s_4) \leq (d_h(s_1, s_3) + d_h(s_2, s_4)) \vee (d_h(s_3, s_2) + d_h(s_1, s_4)). \quad (3)$$

In particular, it implies the triangle inequality by taking $s_3 = s_4$. We introduce the equivalence relation \sim_h defined by $s \sim_h t$ iff $d_h(s, t) = 0$ (or equivalently iff $h(s) = h(t) = m_h(s, t)$). Let T_h be the quotient space

$$T_h = [0, M] / \sim_h.$$

The function d_h induces a distance on T_h that is also denoted by d_h . Thus, (T_h, d_h) is a metric space satisfying the four points conditions. Denote by $p_h : [0, M] \rightarrow T_h$ the canonical projection. p_h is continuous when h is continuous and then (T_h, d_h) is compact and path-connected. Now observe that this construction can be done with any non-negative function on $[0, M]$ but the resulting metric space may not be path-connected (take for instance an increasing function with a unique positive jump). We shall prove in Lemma 2.1 that if $h \in \mathcal{H}_M$, then (T_h, d_h) is a compact real tree.

Observe that the construction of (T_h, d_h) provides interesting additional structures:

- Firstly, the construction provides a special vertex $\rho_h = p_h(0)$ called the *root* of the tree. T_h can be viewed as family tree and the root as the ancestor of the family; it induces a partial order \preceq given by

$$\sigma \preceq \sigma' \quad \text{iff} \quad \sigma \in \llbracket \rho_h, \sigma' \rrbracket.$$

This order is called the *genealogical order* associated with the rooted tree (T_h, d_h, ρ_h) .

- Secondly, the construction provides a relation \leq_h on the tree that is induced by the usual order on $[0, M]$. More precisely,

$$\sigma \leq_h \sigma' \quad \text{iff} \quad \inf p_h^{-1}(\{\sigma\}) \leq \inf p_h^{-1}(\{\sigma'\}).$$

The relation \leq_h is actually a linear (or total) order (antisymmetry is the only non obvious point to prove: to that end use Lemma 2.2). This order is the analogue of the lexicographical order on discrete rooted ordered trees.

- Thirdly, the construction provides a measure μ_h that is the measure on T_h induced by the Lebesgue measure λ on $[0, M]$. More precisely, for any Borel set A in T_h :

$$\mu_h(A) = \lambda(p_h^{-1}(A)).$$

We call a rooted, linearly ordered and measured compact real tree a *structured* compact real tree. In this paper we investigate the problem to know which of the structured compact real trees can be obtained by such a construction. More precisely, we say that two structured compact real trees (T, d, ρ, \leq, μ) and $(T', d', \rho', \leq', \mu')$ are *equivalent* iff there exists an isometry f from (T, d) onto (T', d') that preserves roots (i.e. $f(\rho) = \rho'$), that preserves orders (i.e. $f(\sigma_1) \leq' f(\sigma_2)$ as soon as $\sigma_1 \leq \sigma_2$) and that preserves measures (i.e. $\mu' = \mu \circ f^{-1}$). Let us introduce notation $\sigma \wedge \sigma'$ for the branching point of σ and σ' that is defined by

$$[\![\rho, \sigma \wedge \sigma']\!] = [\![\rho, \sigma]\!] \cap [\![\rho, \sigma']\!] .$$

Let assume that (T, d, ρ, \leq, μ) is equivalent to a structured tree obtained by the coding via a height function h ; of course the order and the measure have to satisfy some compatibility conditions. More precisely, we claim that necessarily, \leq and μ have to satisfy the following conditions:

- **(Or1)** For any σ_1, σ_2 in T , if $\sigma_1 \in [\![\rho, \sigma_2]\!]$, then $\sigma_1 \leq \sigma_2$.
- **(Or2)** If $\sigma_1 \leq \sigma_2 \leq \sigma_3$, then $\gamma \in [\![\rho, \sigma_2]\!]$ where γ stands for the branching point of σ_1 on the subtree spanned by ρ, σ_2 and σ_3 , namely:

$$[\![\rho, \gamma]\!] = [\![\rho, \sigma_1]\!] \cap ([\![\rho, \sigma_2]\!] \cup [\![\rho, \sigma_3]\!]) .$$

(see Figure 1 and Remark 1.1).

- **(Mes)** For any distinct σ_1 and σ_2 in T such that $\sigma_1 < \sigma_2$, we have

$$\mu(\{\sigma \in T : \sigma_1 < \sigma < \sigma_2\}) > 0.$$

This claim shall be proved in Lemma 2.3. A linear order satisfying (Or1) and (Or2) is said to be *compatible* (with the metric and the choice of a root) and a measure satisfying (Mes) is also said to be *compatible* (with the metric, the root and \leq).

Remark 1.1 If \leq satisfies (Or2), then $\sigma_1 \leq \sigma_2 \leq \sigma_3$ implies

$$d(\sigma_1, \sigma_3) \geq d(\sigma_2 \wedge \sigma_3, \sigma_3)$$

and $\gamma = \sigma_1 \wedge \sigma_2$. □

Remark 1.2 Observe that if μ satisfies (Mes), then its topological support $\text{supp}(\mu)$ is T . But the converse is not true: see Figure 2. □

Remark 1.3 We shall describe in Proposition 2.6 all the compatible orders that can be defined on a given rooted compact real tree. We also explain in Section 2.2 that there is a natural way to pick uniformly at random these compatible orders. More precisely, on any fixed rooted

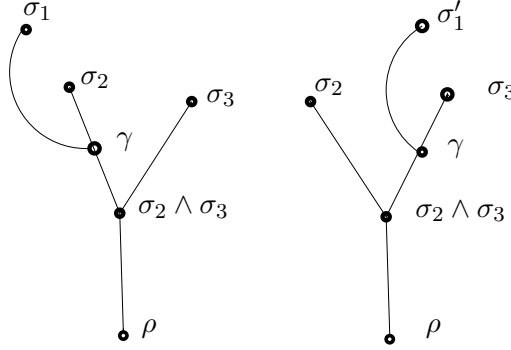


Figure 1: In the first case we can find an order \leq that satisfies (Or2) and $\sigma_1 < \sigma_2 < \sigma_3$, while in the second case, it is not possible to find an order \leq that satisfies (Or2) and $\sigma'_1 < \sigma_2 < \sigma_3$.

compact real tree (T, d, ρ) we shall construct a random compatible order denoted by \leq_{Sh} such that for any $\sigma_1, \dots, \sigma_n$ distinct elements of T , the random ordering induced on $\{\sigma_1, \dots, \sigma_n\}$ by \leq_{Sh} is uniformly distributed among all the distinct orderings of $\{\sigma_1, \dots, \sigma_n\}$ induced by linear orders satisfying (Or1) and (Or2) (see Proposition 2.7 for details). This random order \leq_{Sh} is called the *uniform random shuffling* of (T, d, ρ) . We shall also prove in Proposition 2.8 that a.s. any finite Borel measure μ on T whose topological support is T satisfies (Mes) with respect to $(T, d, \rho, \leq_{\text{Sh}})$. \square

Remark 1.4 Observe that different functions in \mathcal{H} may correspond to the same structured tree: consider for instance $h_1 \in \mathcal{H}_1$ that is the non-decreasing, continuous and piecewise linear height function such that $h_1(1/3) = h_1(2/3) = 1/2$ and $h_1(1) = 1$, and define h_2 by

$$h_2(t) = \mathbf{1}_{[0, 1/2]}(t) h_1(2t) + \mathbf{1}_{(1/2, 1]}(t) h_1(2(1 - t)).$$

See Figure 3. First note that $h_2 \in \mathcal{H}_1$ and then observe that h_1 and h_2 code the same structured compact real tree (T, d, ρ, \leq, μ) where T is $[0, 1]$, d is the usual distance on $[0, 1]$, ρ is 0, \leq is the usual order on the line and $\mu = 2/3\lambda + 1/3\delta_{1/2}$ (here λ stands for the Lebesgue measure on $[0, 1]$ and $\delta_{1/2}$ is the Dirac mass at $1/2$). \square

As in the discrete case where the height process only visits the vertices once in the lexicographical order, we get uniqueness of the coding by requiring that the height function backtracks

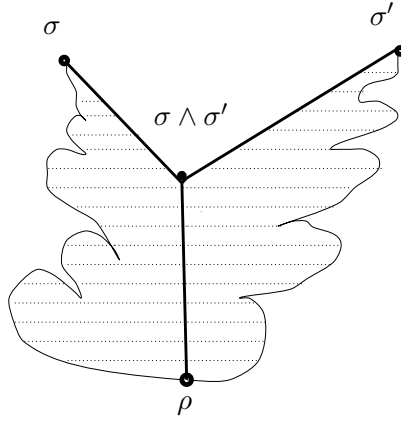


Figure 2: Assume that σ, σ' are two distinct leaves of the compact rooted real tree (T, d, ρ) and assume that a bush is grafted on each vertex of a dense subset of the subtree $T' = \llbracket \rho, \sigma \rrbracket \cup \llbracket \rho, \sigma' \rrbracket$. Thus, $T \setminus T'$ (that is the dashed part of the tree) is dense in T . Then, there exists a finite Borel measure μ whose topological support is T and such that $\mu(T') = 0$. Now assume that the bushes grafted on $\llbracket \sigma \wedge \sigma', \sigma \rrbracket$ are all on the left and that the bushes grafted on $\llbracket \sigma \wedge \sigma', \sigma' \rrbracket$ are all on the right so that $\llbracket \sigma \wedge \sigma', \sigma' \rrbracket = \{\xi \in T : \sigma < \xi < \sigma'\}$. Now, observe that μ does not satisfy (Mes).

as less as possible. More precisely, let $h \in \mathcal{H}$; for any $\sigma \in T_h$, set

$$\ell(\sigma) = \inf p_h^{-1}(\{\sigma\}) \quad \text{and} \quad r(\sigma) = \inf\{t > \ell(\sigma) : p_h(t) \neq \sigma\}.$$

We shall prove in Lemma 2.2 that $p_h(\ell(\sigma)) = \sigma$ and the left-continuity of h implies that $p_h(r(\sigma)) = \sigma$. So, we call the two sets

$$F(h) = \bigcup_{\sigma \in T_h} [\ell(\sigma), r(\sigma)] \quad \text{and} \quad S(h) = [0, \zeta(h)] \setminus F(h)$$

resp. the set of *times of first visit* and the set of *times of latter visit*. We shall prove in Lemma 3.7 that $F(h)$ and $S(h)$ are Borel sets of the real line. We introduce the following property:

- **(Min)** The height function $h \in \mathcal{H}$ is said *minimal* iff $\lambda(S(h)) = 0$.

One of the two main results of the paper is the following.

Theorem 1.1 *Let (T, d, ρ, \leq, μ) be a structured compact real tree such that \leq satisfies (Or1) and (Or2) and such that μ satisfies (Mes). There exists a unique $h \in \mathcal{H}$ satisfying (Min) such that (T, d, ρ, \leq, μ) and $(T_h, d_h, \rho_h, \leq_h, \mu_h)$ are equivalent.*

This theorem is proved in Section 3.

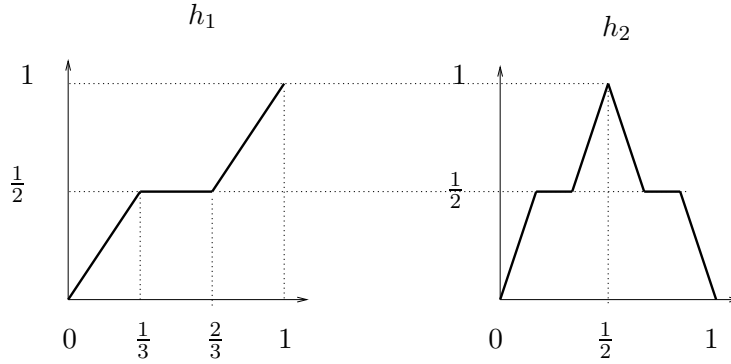


Figure 3: h_1 and h_2 are two functions coding the same structured compact real tree. Observe that $F(h_1) = [0, 1]$ and that $F(h_2) = [0, 1/2]$. Thus, h_1 satisfies (Min) while h_2 does not.

Corollary 1.2 *Let (T, d) be a rooted compact real tree. There exists a continuous function $c \in \mathcal{H}$ such that (T_c, d_c) and (T, d) are isometric.*

Proof of Corollary 1.2: We first fix a root $\rho \in T$. We can always find a probability measure μ whose topological support is T : consider for instance a sequence σ_n , $n \geq 1$, that is dense in (T, d) and define

$$\mu(d\sigma) = \sum_{n \geq 1} 2^{-n} \delta_{\sigma_n}(d\sigma) .$$

As already mentioned in Remark 1.3, we can always find a linear order \leq on T such that the structured tree (T, d, ρ, \leq, μ) satisfies (Or1), (Or2) and (Mes). Denote by h the coding height function associated with (T, d, ρ, \leq, μ) by Theorem 1.1. It implies in particular that (T, d, ρ, \leq) and $(T_h, d_h, \rho_h, \leq_h)$ are equivalent. We conclude thanks to Lemma 4.2 proved in Section 4 that asserts that with any $h \in \mathcal{H}$, we can always associate a (non-unique) continuous $c \in \mathcal{H}$ such that $(T_h, d_h, \rho_h, \leq_h)$ and $(T_c, d_c, \rho_c, \leq_c)$ are equivalent. \blacksquare

Remark 1.5 The result of the corollary has been proved independently by J-F. Le Gall [31] by an approximation procedure. \square

Comment 1.1 Let (T, d, ρ, \leq, μ) be a structured compact real tree such that \leq satisfies (Or1) and (Or2) and such that μ satisfies (Mes). Assume that all the mass of μ is on the leaves of T , namely

$$\mu(\text{Sk}(T)) = 0 . \tag{4}$$

Let $h \in \mathcal{H}$ be such that $(T_h, d_h, \rho_h, \leq_h, \mu_h)$ is equivalent to (T, d, ρ, \leq, μ) . Since

$$p_h^{-1}(\text{Lf}(T)) \subset F(h) ,$$

h automatically satisfies (Min). Consequently, there is a unique height function that codes a structured compact real tree (T, d, ρ, \leq, μ) that satisfies (Or1), (Or2), (Mes) and (4). (Let us mention that it is the case of the Continuum Random Tree that is coded by the normalized Brownian excursion (see [4, 34])).

We have shown in the proof of Corollary 1.2 that it is always possible to find a continuous height function $c \in \mathcal{H}$ that codes (T, d, ρ, \leq) where \leq satisfies (Or1), (Or2). But if we fix μ on T satisfying (Mes), there may be no continuous function coding the structured tree (T, d, ρ, \leq, μ) . For instance, assume that μ satisfies (4) and denote by h the unique height function that codes (T, d, ρ, \leq, μ) ; choose $t \in (0, \zeta(h))$ such that

$$p_h(t) \in \text{Lf}(T_h) \quad \text{and} \quad h(t) > 0.$$

Define $h' \in \mathcal{H}$ by

$$h'(s) = \mathbf{1}_{[0, t]}(s) h(s) + \frac{1}{2} \mathbf{1}_{(t, \zeta(h)]}(s) h(s).$$

It is easy to check that $\mu_{h'}(\text{Sk}(T_{h'})) = 0$. Thus, h' is the unique height function that codes $(T_{h'}, d_{h'}, \rho_{h'}, \leq_{h'}, \mu_{h'})$ and obviously h' is not continuous at t . \square

In view of the previous corollary and of Lemma 4.2 in Section 4, let us note that there are many height functions coding the same compact real tree. More precisely, let $h' \in \mathcal{H}$ and let φ be an increasing continuous mapping from a finite interval $[0, M]$ onto $[0, \zeta(h')]$. Then, observe that the function defined by $h = h' \circ \varphi$ is in \mathcal{H}_M and also observe that the two ordered rooted compact real trees $(T_h, d_h, \rho_h, \leq_h)$ and $(T_{h'}, d_{h'}, \rho_{h'}, \leq_{h'})$ are equivalent. The time-change φ only affects the measures μ_h and $\mu_{h'}$. Of course a measure-change does not always correspond to a re-parametrization of the coding functions: consider for instance h_1 and h_2 as in Remark 1.4 and observe that h_2 cannot be obtained from h_1 by a time-change, while h_1 and h_2 code the same structured tree (T, d, ρ, \leq, μ) . However, if we require that the height functions h and h' both satisfy (Min), then we can have a precise result explained in the following theorem that is proved in Section 3.

Theorem 1.3 *Let (T, d, ρ) be a rooted compact real tree and let \leq be a linear order satisfying (Or1) and (Or2). Let μ and μ' be two finite Borel measures on T that both satisfy (Mes). Denote by h and h' the height functions associated with resp. (T, d, ρ, \leq, μ) and (T, d, ρ, \leq, μ') by Theorem 1.1 (h and h' then satisfy (Min)). Then, there exists a non-decreasing and left-continuous mapping $\varphi : [0, \mu(T)] \rightarrow [0, \infty)$ such that*

$$\varphi(0) = 0, \quad \text{and} \quad h = h' \circ \varphi.$$

Moreover, the following assertions are true:

- (i) The time-change φ is unique iff for any $\sigma \in T$, $\mu(\{\sigma\})\mu'(\{\sigma\}) = 0$, namely iff μ and μ' do not share any atom.
- (ii) If μ' has no atom, then φ is continuous.
- (iii) If μ has no atom, then φ is increasing.

- (iv) For any $t \in [0, \zeta(h))$ such that $\varphi(t) < \varphi(t+)$,

$$h'(u) = h(t), \quad u \in [\varphi(t), \varphi(t+)] .$$

Remark 1.6 Observe that if $\mu(\{\rho\}) = 0$ and $\mu'(\{\rho\}) > 0$, then $\varphi(0+) > 0$. \square

Comment 1.2 If $h, h' \in \mathcal{H}$ are as in Theorem 1.3, then h is continuous iff h' is continuous. As already mentioned, Lemma 4.2 asserts that any ordered compact tree (T, d, ρ, \leq) can be coded by a continuous height function $c \in \mathcal{H}$. But the latter observation implies that in certain cases, it is impossible to find such a continuous function satisfying (Min). Consider for instance a “Y-shaped” tree rooted at the foot of the “Y” (namely a tree with two leaves and a root distinct from the branching point of the leaves) equipped with any of the two compatible orders. This tree cannot be coded by a continuous function satisfying (Min) (see Figure 4). See Section 4 for detailed results about continuity properties of height functions.

More generally, properties of h as a path (i.e. properties that do not depend on any parametrization of h) only concern (T, d, ρ, \leq) . In that vein, we prove in Proposition 4.1 that the total length $\ell_T(T)$ of a compact real tree (T, d) is finite iff there exists a height function h with bounded variation that codes (T, d) . \square

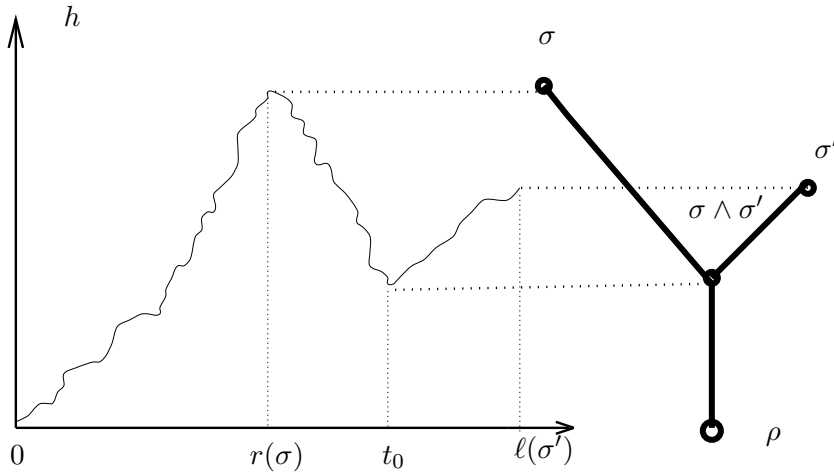


Figure 4: Set $t_0 = \inf\{t \in (r(\sigma), \ell(\sigma')) : h(t) = d_h(\rho, \sigma \wedge \sigma')\}$. Observe that if h is continuous, then $t_0 > r(\sigma)$ but note that necessarily, $(r(\sigma), t_0) \subset S(h)$. Thus, h does not satisfies (Min).

Comment 1.3 Observe that if $h, h' \in \mathcal{H}$ are as in Theorem 1.3 and if μ and μ' have no atom, then there exists a unique increasing and continuous time-change φ mapping $[0, \zeta(h)]$ onto $[0, \zeta(h')]$ and such that $h = h' \circ \varphi$. \square

Comment 1.4 In Aldous's terminology, a *continuum tree* is a rooted compact real tree (T, d, ρ) equipped with a probability measure μ that satisfies the following conditions:

- (CT1) The topological support of μ is T .
- (CT2) The measure μ is non-atomic.
- (CT3) The measure μ is supported by the set of leaves of T , namely $\mu(\text{Sk}(T)) = 0$.

First, let us mention that the definition of a continuum tree given here is slightly different from the definition given by Conditions (a), (b), (c) p. 253 and Assumption (17) p. 265 in [5]: however the difference is inessential for our purpose. Next, observe that (CT1), (CT2), (CT3) imply a certain topological constraint on (T, d, ρ) , namely that the leaves are dense in T :

$$\overline{\text{Lf}(T)} = T. \quad (5)$$

Conversely, if T satisfies (5), then we can find a probability measure μ such that (T, d, ρ, μ) is a continuum tree (See Proposition 4.4).

Let us put the uniform random shuffling \leq_{Sh} on (T, d, ρ) and denote by h_{Sh} the height function associated with $(T, d, \rho, \leq_{\text{Sh}}, \mu)$ by Theorem 1.1. As a direct consequence of Proposition 4.3 and Remark 4.2 in Section 4, a.s. the height function h_{Sh} is continuous and a.s. it is the unique element of \mathcal{H} that codes $(T, d, \rho, \leq_{\text{Sh}}, \mu)$. Moreover, by Theorem 1.3 and Comment 1.3, a.s. for any μ' on T satisfying (CT1), (CT2) and (CT3), the height function h'_{Sh} coding $(T, d, \rho, \leq_{\text{Sh}}, \mu')$ is obtained from h_{Sh} by a unique increasing continuous time-change φ . \square

The paper is organized as follows: in Section 2.1, we describe all the possible linear orders satisfying (Or1) and (Or2); in Section 2.2, we define the uniform random shuffling of a compact rooted real tree; in Section 2.3 we prove topological properties of compatible linear orders. Section 3 is devoted to the proofs of Theorem 1.1 and Theorem 1.3. In the last section, we discuss special properties of height functions and we make the connection with an earlier result of Aldous (namely Theorem 15 in [5]) that provides a randomized construction of height functions in the special case of continuum trees. We conclude Section 4 with a probabilistic example illustrating the effect of order-change on height functions.

2 Compatible linear orders.

2.1 Construction.

Let (T, d, ρ) be a rooted compact real tree. In this section we construct all compatible linear orders on T . But first, let us prove the following proposition.

Lemma 2.1 *For any $h \in \mathcal{H}$, (T_h, d_h) is a compact real tree.*

Proof: Let us prove that if h is in \mathcal{H}_M , then (T_h, d_h) is path-connected. Recall that p_h stands for the canonical projection from $[0, M]$ onto T_h ; note that p_h is not necessary continuous; set $\rho_h = p_h(0)$; let $t_0 \in [0, M]$ and set $\sigma = p_h(t_0)$; for any $s \in [0, h(t_0)]$ define $\sigma(s) = p_h(i(s))$ where $i(s)$ is given by

$$i(s) = \sup\{t \in [0, t_0] : h(t) \leq s\}.$$

Since h is caglad without positive jump, $h(i(s)) = h(i(s)+) = s$. Moreover, it is also easy to check that

$$\sigma(0) = \rho, \sigma(h(t_0)) = \sigma \quad \text{and} \quad d_h(\sigma(s), \sigma(s')) = |s - s'|, \quad s, s' \in [0, h(t_0)]. \quad (6)$$

Thus, (T_h, d_h) is path-connected.

Let us prove now that (T_h, d_h) is compact. Let σ_n , $n \geq 1$, be a T_h -valued sequence; let $t_n \in [0, \zeta(h)]$, $n \geq 1$, be such that $p_h(t_n) = \sigma_n$; we can always find a monotone subsequence t_{k_n} , $n \geq 1$, that converges to $t \in [0, \zeta(h)]$ (say). If t_{k_n} , $n \geq 1$, is non-decreasing, then the left-continuity of h implies that $d_h(t_{k_n}, t)$ goes to 0 and consequently,

$$\lim_{n \rightarrow \infty} d_h(\sigma_{k_n}, p_h(t)) = 0.$$

Assume that t_{k_n} , $n \geq 1$, is non-increasing. By definition $h(t+) \leq h(t)$ and set

$$t' = \sup\{s \in [0, t] : h(s) \leq h(t+)\}.$$

Then observe that $d_h(t_{k_n}, t')$ goes to 0, which implies that

$$\lim_{n \rightarrow \infty} d_h(\sigma_{k_n}, p_h(t')) = 0.$$

Thus, it proves that (T_h, d_h) is compact. Use the four points condition to complete the proof of the lemma. ■

Next let us prove that for any $h \in \mathcal{H}$, the relation \leq_h is an order that satisfies (Or1), (Or2) and that the measure μ_h satisfy (Mes). Recall that

$$\ell(\sigma) = \inf p_h^{-1}(\{\sigma\}).$$

We need the following lemma.

Lemma 2.2 *For any $\sigma \in T_h$, we have $p_h(\ell(\sigma)) = \sigma$.*

Proof: If h is continuous at $\ell(\sigma)$, then the result is obvious. Assume that $h(\ell(\sigma)) > h(\ell(\sigma)+)$ and suppose that $p_h(\ell(\sigma)) \neq \sigma$. Then, by definition of $\ell(\sigma)$, there is a decreasing sequence t_n , $n \geq 1$, converging to $\ell(\sigma)$ such that $p_h(t_n) = \sigma$. Then, $h(t_n) = d(\rho, \sigma)$ for any $n \geq 1$, and

$$\lim_{n \rightarrow \infty} h(t_n) = h(\ell(\sigma)+) = d(\rho, \sigma) < h(\ell(\sigma)).$$

Set $s_0 = \sup\{s \in [0, \ell(\sigma)) : h(s) \leq h(\ell(\sigma)+)\}$. Observe that $m_h(s_0, t_n) = h(\ell(\sigma)+)$, $n \geq 1$. It implies that $p_h(s_0) = p_h(t_n) = \sigma$. But clearly $s_0 < \ell(\sigma)$ which contradicts the definition of $\ell(\sigma)$. Thus, $p_h(\ell(\sigma)) = \sigma$, which completes the proof of the lemma. \blacksquare

This lemma implies that the relation \leq_h is antisymmetric, which is the only non-obvious point to justify in order to prove that \leq_h is a linear order.

Proposition 2.3 *Let $h \in \mathcal{H}$. Then, \leq_h satisfies (Or1) and (Or2) and μ_h satisfies (Mes).*

Proof: Let $\sigma_1, \sigma_2 \in T_h$ be such that $\sigma_1 \in \llbracket \rho_h, \sigma_2 \rrbracket$. Set

$$t_0 = \sup\{0 \leq s \leq \ell(\sigma_2) : h(s) \leq d_h(\rho_h, \sigma_1)\}.$$

Since h is caglad without negative jump, we get $h(t_0) = d_h(\rho_h, \sigma_1) = h(t_0+)$. Thus by the previous lemma

$$m_h(t_0, \ell(\sigma_2)) = h(t_0) = d_h(\rho_h, \sigma_1) < d_h(\rho_h, \sigma_2) = h(\ell(\sigma_2))$$

and $p_h(t_0) = \sigma_1$. Then $\ell(\sigma_1) \leq t_0 < \ell(\sigma_2)$. Thus, $\sigma_1 \leq_h \sigma_2$, which proves that \leq_h satisfies (Or1).

Let us prove that \leq_h satisfies (Or2): let $\sigma_1 \leq_h \sigma_2 \leq_h \sigma_3$, which implies that $\ell(\sigma_1) \leq \ell(\sigma_2) \leq \ell(\sigma_3)$ by definition. By the previous lemma, $p_h(\ell(\sigma_i)) = \sigma_i$, $i \in \{1, 2, 3\}$. Consequently,

$$d_h(\rho_h, \sigma_1 \wedge \sigma_3) = \min_{\ell(\sigma_1) \leq t \leq \ell(\sigma_3)} h(t) \leq \min_{\ell(\sigma_1) \leq t \leq \ell(\sigma_2)} h(t) = d_h(\rho_h, \sigma_1 \wedge \sigma_2),$$

which implies (Or2).

Let us prove that μ_h satisfies (Mes): let $\sigma_1 <_h \sigma_2$. By Lemma 2.2, we get $m_h(\ell(\sigma_1), \ell(\sigma_2)) = d_h(\rho_h, \sigma_1 \wedge \sigma_2)$. The left-continuity of h implies that there exists $t_0 \in [\ell(\sigma_1), \ell(\sigma_2))$ such that

$$\forall t \in [t_0, \ell(\sigma_2)] , \quad h(t) > m_h(\ell(\sigma_1), \ell(\sigma_2)).$$

Choose $t \in (t_0, \ell(\sigma_2)]$ and set $\sigma = p_h(t)$. Observe that for any $s \sim_h t$ and any $s' \in [0, \ell(\sigma_1)]$

$$m_h(s', \ell(\sigma_2)) \leq m_h(\ell(\sigma_1), \ell(\sigma_2)) = d_h(\rho_h, \sigma_1 \wedge \sigma_2) < m_h(t, \ell(\sigma_2)) = m_h(s, \ell(\sigma_2)).$$

It implies that $\inf p_h^{-1}(\{\sigma\}) \geq t_0 \geq \ell(\sigma_1)$ and thus $\sigma_1 <_h \sigma$. Consequently,

$$(t_0, \ell(\sigma_2)) \subset p_h^{-1}(\{\sigma : \sigma_1 <_h \sigma <_h \sigma_2\}),$$

which implies (Mes) since $t_0 < \ell(\sigma_2)$. \blacksquare

Let (T, d, ρ) be a rooted compact real tree. To avoid trivialities, we assume that T is not a point. We now construct a compatible linear order on T . To that end we need to introduce some notation: for any $\sigma \in T$, we denote by \mathcal{C}_σ the set of the connected components of $T \setminus \{\sigma\}$ that do not contain the root ρ . Observe that \mathcal{C}_σ is empty iff σ is a leaf. We also introduce the following set \mathcal{I}

$$\mathcal{I} = \{\emptyset; \mathbb{N}^*; \{1, \dots, n\}, n \in \mathbb{N}^*\}.$$

We think of \mathcal{I} as a family of indexing sets. More precisely, with any $\sigma \in T$ we associate $I_\sigma \in \mathcal{I}$ and a bijection C_σ from I_σ onto \mathcal{C}_σ such that

$$\mathcal{C}_\sigma = \{C_\sigma(k) , k \in I_\sigma\}.$$

Recall that $\text{Br}(T)$ stands for the set of branching points of T and that $\rho \notin \text{Br}(T)$, by convention. For any $\sigma \in \text{Br}(T) \cup \{\rho\}$ choose a linear order on I_σ that is denoted by \triangleleft_σ . Set $\mathbf{O} = \{\triangleleft_\sigma , \sigma \in \text{Br}(T) \cup \{\rho\}\}$. We define a binary relation $\triangleleft_{\mathbf{O}}$ on T in the following way: let σ and σ' be two distinct elements of T .

- **(Def1)** If $\sigma \wedge \sigma' \in \{\sigma, \sigma'\}$ then we set $\sigma \triangleleft_{\mathbf{O}} \sigma'$ if $\sigma \in \llbracket \rho, \sigma' \rrbracket$ and we set $\sigma' \triangleleft_{\mathbf{O}} \sigma$ if $\sigma' \in \llbracket \rho, \sigma \rrbracket$.
- **(Def2)** If $\sigma \wedge \sigma' \notin \{\sigma, \sigma'\}$, then $\sigma \wedge \sigma' \in \text{Br}(T) \cup \{\rho\}$ and there exist two distinct integers k and k' in $I_{\sigma \wedge \sigma'}$ such that $\sigma \in C_{\sigma \wedge \sigma'}(k)$ and $\sigma' \in C_{\sigma \wedge \sigma'}(k')$; then we set $\sigma \triangleleft_{\mathbf{O}} \sigma'$ if $k \triangleleft_{\sigma \wedge \sigma'} k'$ and $\sigma' \triangleleft_{\mathbf{O}} \sigma$ if $k' \triangleleft_{\sigma \wedge \sigma'} k$.

Proposition 2.4 $\triangleleft_{\mathbf{O}}$ is a linear order satisfying (Or1) and (Or2).

Proof: By definition, for any σ, σ' in T , we either have $\sigma \triangleleft_{\mathbf{O}} \sigma'$ or $\sigma' \triangleleft_{\mathbf{O}} \sigma$ so that the relation $\triangleleft_{\mathbf{O}}$ is linear. Let us prove that $\triangleleft_{\mathbf{O}}$ is antisymmetric. Suppose that $\sigma \triangleleft_{\mathbf{O}} \sigma'$ and $\sigma' \triangleleft_{\mathbf{O}} \sigma$; if $\sigma \wedge \sigma' \in \{\sigma, \sigma'\}$ then (Def1) easily implies that $\sigma = \sigma'$; suppose that $\sigma \wedge \sigma' \notin \{\sigma, \sigma'\}$; then with the notation of (Def2), we should have $k \triangleleft_{\sigma \wedge \sigma'} k'$ and $k' \triangleleft_{\sigma \wedge \sigma'} k$, which implies that $k = k'$; thus σ and σ' would be in the same connected component of $T \setminus \{\sigma \wedge \sigma'\}$, which is impossible by definition of $\sigma \wedge \sigma'$.

Let us prove that $\triangleleft_{\mathbf{O}}$ is transitive. Let $\sigma_1, \sigma_2, \sigma_3 \in T$ be such that $\sigma_1 \triangleleft_{\mathbf{O}} \sigma_2$ and $\sigma_2 \triangleleft_{\mathbf{O}} \sigma_3$. To avoid trivialities, we assume that $\sigma_1 \wedge \sigma_3 \notin \{\sigma_1, \sigma_3\}$ and that σ_1 and σ_2 are distinct. Let γ be such that

$$\llbracket \rho, \gamma \rrbracket = \llbracket \rho, \sigma_1 \rrbracket \cap (\llbracket \rho, \sigma_2 \rrbracket \cup \llbracket \rho, \sigma_3 \rrbracket).$$

First assume that $\sigma_1 \wedge \sigma_2 \in \llbracket \sigma_1 \wedge \sigma_3, \sigma_1 \rrbracket$. Then $\sigma_2 \wedge \sigma_3 = \sigma_1 \wedge \sigma_3$ and σ_1 and σ_2 are in the same connected component of $\mathcal{C}_{\sigma_1 \wedge \sigma_3}$. There exist two distinct integers k and k' in $I_{\sigma_1 \wedge \sigma_3}$ such that

$$\sigma_1, \sigma_2 \in C_{\sigma_1 \wedge \sigma_3}(k) \quad \text{and} \quad \sigma_3 \in C_{\sigma_1 \wedge \sigma_3}(k').$$

By definition, $\sigma_2 \triangleleft_{\mathbf{O}} \sigma_3$ implies $k \triangleleft_{\sigma_1 \wedge \sigma_3} k'$. Since $\sigma_2 \wedge \sigma_3 = \sigma_1 \wedge \sigma_3$, we also get $\sigma_1 \triangleleft_{\mathbf{O}} \sigma_3$. In addition, observe that $\gamma = \sigma_1 \wedge \sigma_2$ so that (Or2) is verified.

If we assume next that $\sigma_2 \wedge \sigma_3 \in \llbracket \sigma_1 \wedge \sigma_3, \sigma_3 \rrbracket$, then we can show by similar arguments that $\sigma_1 \triangleleft_{\mathbf{O}} \sigma_3$ and that $\gamma \in \llbracket \rho, \sigma_2 \rrbracket$.

It remains to consider the case $\sigma_2 \wedge \sigma_3 \in \llbracket \rho, \sigma_1 \wedge \sigma_3 \rrbracket$: if $\sigma_2 = \sigma_2 \wedge \sigma_3$, then $\sigma_2 \in \llbracket \rho, \sigma_1 \rrbracket$. By (Def1), it implies $\sigma_2 \triangleleft_{\mathbf{O}} \sigma_1$; we have shown that it implies $\sigma_2 = \sigma_1$ which contradicts the assumption that σ_1 and σ_2 are distinct. Thus, $\sigma_2 \neq \sigma_2 \wedge \sigma_3$. Consequently, σ_1 and σ_3 are in the same connected component of $\mathcal{C}_{\sigma_2 \wedge \sigma_3}$; so there exist two distinct integers k and k' in $I_{\sigma_2 \wedge \sigma_3}$ such that

$$\sigma_1, \sigma_3 \in C_{\sigma_2 \wedge \sigma_3}(k) \quad \text{and} \quad \sigma_2 \in C_{\sigma_2 \wedge \sigma_3}(k').$$

But $\sigma_2 \triangleleft_{\mathbf{O}} \sigma_3$ implies $k \triangleleft_{\sigma_2 \wedge \sigma_3} k'$ and $\sigma_1 \triangleleft_{\mathbf{O}} \sigma_2$ implies $k' \triangleleft_{\sigma_2 \wedge \sigma_3} k$, which rises a contradiction. Thus, we cannot have $\sigma_2 \wedge \sigma_3 \in \llbracket \rho, \sigma_1 \wedge \sigma_3 \rrbracket$.

We have proved that $\triangleleft_{\mathbf{O}}$ is a linear order satisfying (Or2). Observe now that (Or1) is a direct consequence of (Def1), which completes the proof of the proposition. ■

Consider now a compatible \leq linear order on (T, d, ρ) . Let $\sigma_0 \in T$ be such that $T \setminus \{\sigma_0\}$ has at least two connected components C and C' that do not contain the root ρ .

Lemma 2.5 *Either $\sigma \leq \sigma'$ for any $\sigma \in C$ and any $\sigma' \in C'$, which is denoted by $C \leq C'$; either $\sigma' \leq \sigma$ for any $\sigma \in C$ and any $\sigma' \in C'$, which is denoted by $C' \leq C$.*

Proof: Suppose that we can find $\sigma_1, \sigma_3 \in C$ and $\sigma_2 \in C'$ such that $\sigma_1 < \sigma_2 < \sigma_3$. Let γ be as in (Or2). We have $\sigma_1 \wedge \sigma_3 \in C$; it implies $\gamma \in C$. Thus $\gamma \notin \llbracket \rho, \sigma_2 \rrbracket$, which contradicts (Or2). Exchange the role of C and C' in the previous argument to complete the proof of the lemma. ■

The lemma implies that for any $\sigma \in \text{Br}(T) \cup \{\rho\}$, we can find a linear order \triangleleft_{σ} on I_{σ} such that for any $k, l \in I_{\sigma}$ that satisfy $k \triangleleft_{\sigma} l$, we have $C_{\sigma}(k) \leq C_{\sigma}(l)$. Consequently, we have proved the following proposition.

Proposition 2.6 *Any compatible linear order on (T, d, ρ) is of the form $\triangleleft_{\mathbf{O}}$, where*

$$\mathbf{O} = \{\triangleleft_{\sigma} , \sigma \in \text{Br}(T) \cup \{\rho\}\}$$

stands for a certain choice of linear orders on the I_{σ} 's , $\sigma \in \text{Br}(T) \cup \{\rho\}$. Moreover, this representation of \leq is unique.

As a consequence of this proposition, two compatible linear orders are obtained one from another by re-ordering each set of indices I_{σ} , $\sigma \in \text{Br}(T) \cup \{\rho\}$.

2.2 Uniform random shuffling of real trees.

In this section we explain how to “pick” a compatible linear order uniformly at random. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which all the random variables that we consider are defined. Let S be a set. We formally define a random order \leq on S as a random mapping $F : \Omega \times (S \times S) \rightarrow \{0, 1\}$ such that $F(\omega; \sigma, \sigma') = \mathbf{1}_{\{\sigma \leq \sigma'\}}(\omega)$ and such that $\omega \rightarrow F(\omega; \cdot, \cdot)$ is $(\mathcal{F}, \mathcal{G})$ -measurable, where we have set

$$\mathcal{G} = \mathcal{E}^{\otimes (S \times S)} \quad \text{with} \quad \mathcal{E} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\} .$$

Example 2.1 Consider for instance $S = \mathbb{N}$ and denote by \mathcal{O} the random uniform order defined by the following property: for any k_1, \dots, k_n distinct elements of \mathbb{N} , the random ordering on $\{k_1, \dots, k_n\}$ induced by \mathcal{O} is uniformly distributed among the $n!$ possible ones. \mathcal{O} is unique in distribution and it can be constructed as follows: let U_n , $n \in \mathbb{N}$ be i.i.d. random variables that are uniformly distributed on $[0, 1]$; we set $n \mathcal{O} m$ iff $U_n \leq U_m$, $m, n \in \mathbb{N}$, where \leq stands here for the usual order on $[0, 1]$ (see Lemma 10 in [5]). □

We shall give a similar construction for the uniform random compatible order on a fixed rooted compact real tree (T, d, ρ) called the *shuffling of T* : Recall that $\text{Br}(T)$ is at most countable; let

$\{U_{\sigma,k}; \sigma \in \text{Br}(T) \cup \{\rho\}, k \in I_\sigma\}$ be a (countable) family of i.i.d. random variables that are uniformly distributed on $[0, 1]$. Define $\mathbf{O} = \{\triangleleft_\sigma, \sigma \in \text{Br}(T) \cup \{\rho\}\}$ by

$$k \triangleleft_\sigma l \quad \text{iff} \quad U_{\sigma,k} \leq U_{\sigma,l}.$$

We define the random uniform shuffling of T by $\leq_{\text{Sh}} = \triangleleft_{\mathbf{O}}$.

Proposition 2.7 *For any $\sigma_1, \dots, \sigma_n$ distinct elements of T , the random ordering of the set $\{\sigma_1, \dots, \sigma_n\}$ induced by \leq_{Sh} is uniformly distributed among the orderings of this set induced by linear orders satisfying (Or1) and (Or2).*

Proof: This is a consequence of the construction of $\triangleleft_{\mathbf{O}}$ given in the previous section and of the result asserted in Example 2.1. The details are left to the reader. \blacksquare

Observe that this proposition implies that \leq_{Sh} is unique in distribution. We now prove the following proposition:

Proposition 2.8 *Almost surely, any finite Borel measure μ whose topological support is T satisfies (Mes) with respect to \leq_{Sh} .*

Proof: Let μ be a finite Borel measure whose topological support is T and let σ_1 and σ_2 be such that $\sigma_2 \notin \llbracket \rho, \sigma_1 \rrbracket$. Thus by definition of \leq_{Sh} , $\mathbb{P}(\sigma_1 <_{\text{Sh}} \sigma_2) \geq 1/2$. Let us fix $\omega_0 \in \{\omega \in \Omega : \sigma_1 <_{\text{Sh}} \sigma_2\}$. There are two cases to consider: suppose first that $\mu(\llbracket \sigma_1 \wedge \sigma_2, \sigma_2 \rrbracket) > 0$; then since we have fixed $\omega_0 \in \{\omega \in \Omega : \sigma_1 <_{\text{Sh}} \sigma_2\}$, we get

$$\llbracket \sigma_1 \wedge \sigma_2, \sigma_2 \rrbracket \subset \{\sigma \in T : \sigma_1 <_{\text{Sh}} \sigma <_{\text{Sh}} \sigma_2\},$$

which implies that $\mu(\{\sigma \in T : \sigma_1 \leq_{\text{Sh}} \sigma \leq_{\text{Sh}} \sigma_2\})$ is non-zero.

Let us now suppose that $\mu(\llbracket \sigma_1 \wedge \sigma_2, \sigma_2 \rrbracket) = 0$. Since the topological support of μ is T , there exists a sequence s_n , $n \geq 1$, of branching points in $\llbracket \sigma_1 \wedge \sigma_2, \sigma_2 \rrbracket$ that is dense in this set; then for any $n \geq 1$, denote by $k_n \in I_{s_n}$ the index such that $\sigma_2 \in C_{s_n}(k_n)$. Fix $n \geq 1$, and take $\sigma \in C_{s_n}(k)$ with $k \in I_{s_n} \setminus \{k_n\}$. Suppose that $\sigma \leq_{\text{Sh}} \sigma_2$. Then, Lemma 2.5 implies that $C_{s_n}(k) \leq_{\text{Sh}} C_{s_n}(k_n)$ and since we have fixed $\omega_0 \in \{\omega \in \Omega : \sigma_1 <_{\text{Sh}} \sigma_2\}$, (Or2) implies that $\sigma_1 <_{\text{Sh}} C_{s_n}(k)$. Consequently

$$\mu(\{\sigma \in T : \sigma_1 <_{\text{Sh}} \sigma <_{\text{Sh}} \sigma_2\}) \geq \mu(C_{s_n}(k)) > 0.$$

Thus, if we fix $\omega_0 \in \{\omega \in \Omega : \sigma_1 <_{\text{Sh}} \sigma_2\}$ and if $\mu(\{\sigma \in T : \sigma_1 <_{\text{Sh}} \sigma <_{\text{Sh}} \sigma_2\}) = 0$, then

- (a) $\mu(\llbracket \sigma_1 \wedge \sigma_2, \sigma_2 \rrbracket) = 0$.
- (b) The denumerable set

$$\{s_n, n \geq 1\} = \text{Br}(T) \cap \llbracket \sigma_1 \wedge \sigma_2, \sigma_2 \rrbracket$$

is dense in $\llbracket \sigma_1 \wedge \sigma_2, \sigma_2 \rrbracket$.

- (c) For any $n \geq 1$, and any $k \in I_{s_n} \setminus \{k_n\}$, we have $C_{s_n}(k_n) \leq_{\text{Sh}} C_{s_n}(k)$. Thus, by definition of \leq_{Sh} , it implies that

$$U_{s_n, k_n}(\omega_0) < U_{s_n, k}(\omega_0), \quad k \in I_{s_n} \setminus \{k_n\}, \quad n \geq 1.$$

Let $\sigma, \sigma' \in T$ such that $\sigma \in \llbracket \rho, \sigma' \rrbracket$; for any $s \in \text{Br}(T) \cap \llbracket \sigma, \sigma' \rrbracket$, denote by $k(s) \in I_s$ the index such that $\sigma' \in C_s(k(s))$. Define

$$A_{\sigma, \sigma'} = \bigcap_{s \in \text{Br}(T) \cap \llbracket \sigma, \sigma' \rrbracket} \{\omega \in \Omega : U_{s, k(s)}(\omega) < U_{s, k}(\omega), \quad k \in I_s \setminus \{k(s)\}\},$$

with the convention $A_{\sigma, \sigma'} = \emptyset$ if $\text{Br}(T) \cap \llbracket \sigma, \sigma' \rrbracket = \emptyset$. Clearly, if $\# \text{Br}(T) \cap \llbracket \sigma, \sigma' \rrbracket = \infty$, then $\mathbb{P}(A_{\sigma, \sigma'}) = 0$. Thus, if we set

$$B = \bigcup \{A_{\sigma, \sigma'} : \sigma, \sigma' \in \text{Br}(T) \cup \{\rho\} : \sigma \in \llbracket \rho, \sigma' \rrbracket \text{ and } \# \text{Br}(T) \cap \llbracket \sigma, \sigma' \rrbracket = \infty\},$$

then $\mathbb{P}(B) = 0$ since $\text{Br}(T)$ is at most countable. Now observe that (a), (b) and (c) imply that

$$\{\omega \in \Omega : \sigma_1 <_{\text{Sh}} \sigma_2\} \cap \{\omega \in \Omega : \mu(\{\sigma \in T : \sigma_1 \leq_{\text{Sh}} \sigma \leq_{\text{Sh}} \sigma_2\}) = 0\} \subset B,$$

which implies the proposition since B does not depend on μ, σ_1 or σ_2 . ■

2.3 Topological properties of compatible linear orders.

In this section we prove properties of compatible linear orders that shall be needed in the next section. Let \leq be a compatible linear order on the rooted compact tree (T, d, ρ) . By Proposition 2.6, \leq is of the form $\triangleleft_{\mathbf{O}}$, for a certain choice $\mathbf{O} = \{\triangleleft_{\sigma}, \sigma \in \text{Br}(T) \cup \{\rho\}\}$ of linear orders on the indexing sets I_{σ} , $\sigma \in \text{Br}(T) \cup \{\rho\}$.

Let us first introduce some notation. Fix $\sigma \in T$. Recall that $\mathcal{C}_{\sigma} = \{C_{\sigma}(k), k \in I_{\sigma}\}$ stands for the set of the connected components of $T \setminus \{\sigma\}$ that do not contain the root. Consider the connected components of $T \setminus \llbracket \rho, \sigma \rrbracket$ that are grafted on $\llbracket \rho, \sigma \rrbracket$: by Lemma 2.5, either all the points of a such component are smaller than σ , either all points of the connected component are greater than σ . So we denote by $\mathcal{C}_{\sigma}^{-} = \{C_j^{-}, j \in J_{\sigma}^{-}\}$ the set of connected components grafted on $\llbracket \rho, \sigma \rrbracket$ that are smaller than σ and we denote by $\mathcal{C}_{\sigma}^{+} = \{C_j^{+}, j \in J_{\sigma}^{+}\}$ the set of connected components grafted on $\llbracket \rho, \sigma \rrbracket$ that are greater than σ . For any $j \in J_{\sigma}^{\pm}$ we denote by γ_j^{\pm} the point of $\llbracket \rho, \sigma \rrbracket$ on which the component C_j^{\pm} is grafted. Observe that $\{\gamma_j^{\pm}\} \cup C_j^{\pm}$ is the closure of C_j^{\pm} . Note that different components may be grafted on the same point. We thus get

$$T \setminus \llbracket \rho, \sigma \rrbracket = \bigcup_{C \in \mathcal{C}_{\sigma}^{-} \cup \mathcal{C}_{\sigma} \cup \mathcal{C}_{\sigma}^{+}} C.$$

Lemma 2.9 *The following assertions are true.*

- For any $j_1 \in J_{\sigma}^{-}$, any $k \in I_{\sigma}$ and any $j_2 \in J_{\sigma}^{+}$, we get

$$C_{j_1}^{-} \leq C_{\sigma}(k) \leq C_{j_2}^{+}.$$

- For any $j_1, j_2 \in J_\sigma^-$ such that $d(\rho, \gamma_{j_1}^-) < d(\rho, \gamma_{j_2}^-)$, we get

$$C_{j_1}^- \leq C_{j_2}^-.$$

- For any $j_1, j_2 \in J_\sigma^+$ such that $d(\rho, \gamma_{j_1}^+) > d(\rho, \gamma_{j_2}^+)$, we get

$$C_{j_1}^+ \leq C_{j_2}^+.$$

Proof: This is a direct consequence of (Def1) and (Def2). ■

We consider the family of subsets of T denoted by L_σ , $\sigma \in T$, and defined by

$$L_\sigma = \{\sigma' \in T : \sigma' \leq \sigma\}.$$

These subsets are called the *left sets* of T . We first prove the following proposition.

Proposition 2.10 *For any $\sigma \in T$, L_σ is a compact set.*

Proof: Observe that

$$T \setminus L_\sigma = \left(\bigcup_{C \in \mathcal{C}_\sigma} C \right) \cup \left(\bigcup_{C^+ \in \mathcal{C}_\sigma^+} C^+ \right).$$

Thus $T \setminus L_\sigma$ is an open set, which proves the proposition. ■

Proposition 2.11 *Every (\leq) -monotone sequence in T is convergent.*

Proof: Let $\sigma_n \in T$, $n \geq 1$, be a (\leq) -monotone sequence. Suppose that it has two distinct limit points σ and σ' . Assume that $\sigma < \sigma'$ and choose $\sigma_0 \in \llbracket \sigma \wedge \sigma', \sigma' \rrbracket$. Denote by C and C' the two distinct connected components of $T \setminus \{\sigma_0\}$ that contain respectively σ and σ' . Note that C also contains the root ρ . Since the sequence is monotone, we can find $n_1, n_2, n_3 \geq 1$, such that

$$\sigma_{n_1} < \sigma_{n_2} < \sigma_{n_3} \quad , \quad \sigma_{n_1}, \sigma_{n_3} \in C' \quad \text{and} \quad \sigma_{n_2} \in C.$$

It first implies that $\sigma_{n_1} \wedge \sigma_{n_3} \in C'$. Then observe that $\llbracket \rho, \sigma_{n_2} \rrbracket \subset C$, which implies that $\sigma_{n_2} \wedge \sigma_{n_3} \in C$ and which contradicts (Or2). Thus, the sequence has only one limit point. ■

We first consider a non-decreasing sequence $\sigma_n \in T$, $n \geq 1$, that converges to σ . We distinguish three cases:

- **Case (I):** $\sigma_n < \sigma$, for all $n \geq 1$.
- **Case (II):** $\sigma_n = \sigma$, for all sufficiently large $n \geq 1$.
- **Case (III):** $\sigma < \sigma_n$, for all sufficiently large $n \geq 1$.

We also set $D = \{\sigma' \in T : \sigma_n < \sigma', n \geq 1\}$. We prove the following lemma.

Lemma 2.12 *The following assertions are true.*

- In Case (I), we get

$$\bigcup_{n \geq 1} L_{\sigma_n} = L_\sigma \setminus \{\sigma\} \quad \text{and} \quad \bigcap_{\sigma' \in D} L_{\sigma'} = L_\sigma.$$

- In Case (II), we get

$$\bigcup_{n \geq 1} L_{\sigma_n} = L_\sigma = \bigcap_{\sigma' \in D} L_{\sigma'}.$$

- In Case (III), we get $\sigma \notin \text{Lf}(T)$ and

$$\bigcup_{n \geq 1} L_{\sigma_n} = L_\sigma \cup \left(\bigcup_{k \in K} C_\sigma(k) \right) = \bigcap_{\sigma' \in D} L_{\sigma'},$$

where we have set

$$K = \{k \in I_\sigma : \exists n \geq 1, \exists l \in I_\sigma : \sigma_n \in C_\sigma(l) \text{ and } k \triangleleft_\sigma l\}$$

if $\sigma \in \text{Br}(T) \cup \{\rho\}$ and $K = I_\sigma = \{1\}$ if $\mathbf{n}(\sigma) = 2$.

Proof: Let us first consider Case (I): suppose that there exists σ' such that $\sigma_n < \sigma' < \sigma$ for all $n \geq 1$. Define γ_n by

$$[\![\rho, \gamma_n]\!] = [\![\rho, \sigma_n]\!] \cap ([\![\rho, \sigma']\!] \cup [\![\rho, \sigma]\!]).$$

Then we get $\gamma_n \in [\![\rho, \sigma']\!]$ by (Or2), which implies that $d(\gamma_n, \sigma) \geq d(\sigma \wedge \sigma', \sigma)$. Thus, for any $n \geq 1$,

$$d(\sigma_n, \sigma) = d(\sigma_n, \gamma_n) + d(\gamma_n, \sigma) \geq d(\sigma \wedge \sigma', \sigma) > 0,$$

which rises a contradiction. This prove the first point of the lemma.

The first equality in Case (II) is obvious. Suppose there exists $\sigma'' \in \bigcap_{\sigma' \in D} L_{\sigma'} \setminus L_\sigma$. Then, σ'' is a minimal element of D . But we can always find $\xi \in T$ such that $\sigma < \xi < \sigma''$. It implies that ξ is also a minimal element of D distinct from σ'' , which is absurd since \leq is linear. It proves the second equality in Case (II).

Let us consider Case (III): We first suppose that there are $n_0 \geq 1$, and $j_0 \in J_\sigma^+$ such that $\sigma_{n_0} \in C_{j_0}^+$. Recall that $\gamma_{j_0}^+$ stands for the point of $[\![\rho, \sigma]\!]$ on which the connected component $C_{j_0}^+$ is grafted. Lemma 2.9 implies that for any $n \geq n_0$, σ_n is in a connected component of \mathcal{C}_σ^+ that is grafted on a point of $[\![\rho, \gamma_{j_0}^+]\!]$. Thus, for all $n \geq n_0$

$$d(\sigma_n, \sigma) > d(\gamma_{j_0}^+, \sigma) > 0,$$

which rises a contradiction. It shows that the sequence σ_n , $n \geq 1$, has no term in any of the connected components of \mathcal{C}_σ^+ . It implies that $\sigma \notin \text{Lf}(T)$ and that

$$\bigcup_{n \geq 1} L_{\sigma_n} \subset L_\sigma \cup \left(\bigcup_{k \in I_\sigma} C_\sigma(k) \right).$$

By similar arguments we also get

$$\bigcup_{n \geq 1} L_{\sigma_n} \subset L_\sigma \cup \left(\bigcup_{k \in K} C_\sigma(k) \right).$$

We now prove the reversed inclusion: suppose that there are $k_0 \in I_\sigma$, $n_0 \geq 1$ and $\sigma' \in C_\sigma(k_0)$ such that

$$\sigma_{n_0} \in C_\sigma(k_0) \quad \text{and} \quad \sigma_n < \sigma', \quad n \geq 1.$$

Then, $\sigma_n \in C_\sigma(k_0)$ for any $n \geq n_0$ and by definition of K we get $K = \{k \in I_\sigma : k \triangleleft_\sigma k_0\}$. Since $\sigma_{n_0} \leq \sigma_n < \sigma'$, (Or2) implies that $\sigma_{n_0} \wedge \sigma' \in \llbracket \sigma, \sigma_n \wedge \sigma' \rrbracket$ and consequently

$$d(\sigma_n, \sigma) \geq d(\sigma_n \wedge \sigma', \sigma) \geq d(\sigma_{n_0} \wedge \sigma', \sigma) > 0,$$

which rises a contradiction. So, it proves that if $\sigma_{n_0} \in C_\sigma(k_0)$, then $C_\sigma(k_0) \subset \bigcup_{n \geq 1} L_{\sigma_n}$. This implies

$$L_\sigma \cup \left(\bigcup_{k \in K} C_\sigma(k) \right) \subset \bigcup_{n \geq 1} L_{\sigma_n},$$

and the first equality of Case (III) follows. Let us prove the second equality of Case (III): clearly, we have

$$\bigcup_{n \geq 1} L_{\sigma_n} \subset \bigcap_{\sigma' \in D} L_{\sigma'}. \quad (7)$$

Suppose now that there is a point σ'' in $\bigcap_{\sigma' \in D} L_{\sigma'}$ such that $\sigma_n < \sigma''$ for all $n \geq 1$. It implies that $\sigma'' \in D$. Since σ'' is in $\bigcap_{\sigma' \in D} L_{\sigma'}$, we have $\sigma'' \leq \sigma'$, $\sigma' \in D$. σ'' is the minimal point of D (there is at most one since the order is linear). Thus, for any $\xi \in T$, if $\xi < \sigma''$, then $\xi \notin D$ and, by definition, there exists $n_0 \geq 1$, such that $\xi \leq \sigma_{n_0}$. This implies

$$\bigcup_{n \geq 1} L_{\sigma_n} = L_{\sigma''} \setminus \{\sigma''\}. \quad (8)$$

But the first equality of Case (III) implies that $\bigcup_{n \geq 1} L_{\sigma_n}$ has to be a compact set, which contradicts (8). So it proves that there is no point σ'' in $\bigcap_{\sigma' \in D} L_{\sigma'}$ such that $\sigma_n < \sigma''$ for all $n \geq 1$. This, combined with (7), implies

$$\bigcup_{n \geq 1} L_{\sigma_n} = \bigcap_{\sigma' \in D} L_{\sigma'},$$

which completes the proof of the Lemma. ■

Consider now a non-increasing sequence $\sigma_n \in T$, $n \geq 1$ that converges to σ . Set

$$G = \{\sigma' \in T : \sigma' < \sigma_n, n \geq 1\}.$$

We prove the following lemma.

Lemma 2.13 *We have*

$$\bigcap_{n \geq 1} L_{\sigma_n} = L_\sigma \cup \left(\bigcup_{k \in I_\sigma \setminus K} C_\sigma(k) \right),$$

where we have set

$$K = \{k \in I_\sigma : \exists n \geq 1, \exists l \in I_\sigma : \sigma_n \in C_\sigma(l) \text{ and } l \triangleleft_\sigma k\}$$

if $\sigma \in \text{Br}(T) \cup \{\rho\}$, $K = I_\sigma = \emptyset$ if $\sigma \in \text{Lf}(T)$ and $K = I_\sigma = \{1\}$ if $n(\sigma) = 2$ (observe that $I_\sigma \setminus K$ may be empty). Moreover, if G is non-empty and if $\sigma_n = \sigma$ for all sufficiently large n , then we get

$$\bigcup_{\sigma' \in G} L_{\sigma'} = L_\sigma \setminus \{\sigma\} \subset L_\sigma = \bigcap_{n \geq 1} L_{\sigma_n}.$$

Otherwise, we get

$$\bigcup_{\sigma' \in G} L_{\sigma'} = \bigcap_{n \geq 1} L_{\sigma_n}.$$

Proof: The arguments are similar to those used to prove Lemma 2.12. The details are left to the reader. ■

We shall need the following lemma in Section 3.

Lemma 2.14 *The collection of sets $\{\emptyset; T; L_\sigma, \sigma \in T\}$ is a π -system that generates the Borel sigma-field of T .*

Proof: Clearly, $\{L_\sigma, \sigma \in T\}$ is closed under finite intersection. This implies that

$$\{\emptyset; T; L_\sigma, \sigma \in T\}$$

is a π -system. Denote by \mathcal{A} the sigma-field generated by this π -system. Let $\sigma \in T$. Deduce from Lemma 2.12 that any connected component of $T \setminus \{\sigma\}$ is in \mathcal{A} (the details are left to the reader). Let $r > 0$ and denote by $\overline{B}(\sigma, r)$ the closed ball with radius r and with center σ . Then,

$$T \setminus \overline{B}(\sigma, r) = \{\sigma' \in T : d(\sigma, \sigma') > r\}.$$

Let C be a connected component of $T \setminus \overline{B}(\sigma, r)$. There exists σ_0 such that $d(\sigma, \sigma_0) = r$ and such that $C \cup \{\sigma_0\}$ is the closure of C . It implies that C is a connected component of $T \setminus \{\sigma_0\}$. Thus, any connected component C of $T \setminus \overline{B}(\sigma, r)$ is in \mathcal{A} . So is $T \setminus \overline{B}(\sigma, r)$, for any $\sigma \in T$ and any $r > 0$, which easily completes the proof of the lemma. ■

3 Construction of the height function.

In this section we prove Theorem 1.1 and Theorem 1.3. Let us consider a rooted compact real tree (T, d, ρ) equipped with a compatible linear order \leq and a compatible measure μ . To avoid trivialities, we assume that T is not a point. Observe that (Mes) implies

- **(Inc):** For any $\sigma < \sigma'$ we get

$$\mu(L_\sigma) < \mu(L_{\sigma'}) .$$

Let us set $M = \mu(T)$. Observe that

$$M = \sup\{\mu(L_\sigma) , \sigma \in T\}.$$

Since $T \neq \{\rho\}$, M is positive. For any $t \in [0, M]$, we define

$$G_t = \{\sigma \in T : \mu(L_\sigma) \leq t\}.$$

We use the following notation:

$$m_t = \sup_{\sigma \in G_t} \mu(L_\sigma) \quad \text{and} \quad M_t = \inf_{\sigma \in T \setminus G_t} \mu(L_\sigma),$$

with the convention that $m_t = 0$ if $G_t = \emptyset$ and that $M_t = M$ if $G_t = T$. Observe that $G_t = T$ iff $t = M$. Clearly, $m_t \leq t \leq M_t$. We also introduce

$$D_t = \bigcap_{\sigma \in T \setminus G_t} L_\sigma ,$$

if $t < M$ and $D_t = T$ if $t = M$. Here is the key lemma used in the proof of Theorem 1.1.

Lemma 3.1 *Fix $t \in [0, M]$. The following assertions are true.*

- (i) $\mu(G_t) = m_t$ and $\mu(D_t) = M_t$.
- (ii) There exist σ_- and σ_+ such that $\sigma_+ \in \llbracket \rho, \sigma_- \rrbracket$ and such that
 - for any non-decreasing sequence σ_n^- , $n \geq 1$, that satisfies

$$\lim_{n \rightarrow \infty} \mu(L_{\sigma_n^-}) = m_t$$

one has

$$\lim_{n \rightarrow \infty} d(\sigma_n^-, \sigma_-) = 0 \quad \text{and} \quad G_t \setminus \{\sigma_-\} \subset \bigcup_{n \geq 1} L_{\sigma_n^-} \subset G_t ;$$

- for any decreasing sequence σ_n^+ , $n \geq 1$, that satisfies

$$\lim_{n \rightarrow \infty} \mu(L_{\sigma_n^+}) = M_t$$

one has

$$\lim_{n \rightarrow \infty} d(\sigma_n^+, \sigma_+) = 0 \quad \text{and} \quad \bigcap_{n \geq 1} L_{\sigma_n^+} = D_t .$$

- (iii) We also get

$$D_t \setminus \{\sigma_-\} \subset G_t \subset D_t.$$

Remark 3.1 There is no non-decreasing sequence σ_n^- , $n \geq 1$ satisfying the condition of (ii) iff $G_t = \emptyset$. In this case, we set $\sigma_- = \rho$, by convention. Note also that there is no decreasing sequence σ_n^+ , $n \geq 1$ satisfying the condition of (ii) iff $M_t = M$. In this case, we set $\sigma_+ = \sigma_-$, by convention.

Proof: First note that $G_t \subset D_t$. Suppose that there exists $\tilde{\sigma} \in D_t \setminus G_t$: then, by definition of D_t , for any $\sigma' \in T \setminus G_t$, we get $\tilde{\sigma} \leq \sigma'$. Consequently, $\tilde{\sigma}$ is the minimal element of $T \setminus G_t$. Observe next that (Inc) implies that $\sigma < \tilde{\sigma}$ for any $\sigma \in G_t$. Thus we have proved that

- either $T \setminus G_t$ has no minimal point and then $G_t = D_t$,
- either $T \setminus G_t$ has a minimal point denoted by $\tilde{\sigma}$ and then

$$G_t = L_{\tilde{\sigma}} \setminus \{\tilde{\sigma}\} \quad \text{and} \quad D_t = L_{\tilde{\sigma}}. \quad (9)$$

We distinguish several cases in the proof:

- *Case 1:* Suppose that there exists $\sigma_* \in G_t$ such that $m_t = \mu(L_{\sigma_*})$. Then, by (Inc) we get $G_t = L_{\sigma_*}$. We first claim that

$$D_t = G_t = L_{\sigma_*}. \quad (10)$$

Clearly $L_{\sigma_*} \subset D_t$. Suppose that there exists $\sigma' \in D_t \setminus L_{\sigma_*}$. Then, σ' is the minimal element of $T \setminus G_t$. Then by (9), G_t is not compact, which rises a contradiction. Then it implies (10) and (i) and (iii) follow.

Let σ_n^- , $n \geq 1$ be as in the lemma. By Lemma 2.11, it has a limit that we denote by σ_- . We claim that

$$\sigma_- = \sigma_*. \quad (11)$$

Indeed, by Lemma 2.12, σ_- is in the closure of $\bigcup_{n \geq 1} L_{\sigma_n^-}$ and since $\sigma_n^- \in L_{\sigma_*}$ for any $n \geq 1$, we get $\sigma_- \in L_{\sigma_*}$. Suppose that we are in Case (I), Case (II) or Case (III) with $\sigma_- \notin \text{Br}(T) \cup \{\rho\}$ of Lemma 2.12. Then, the closure of $\bigcup_{n \geq 1} L_{\sigma_n^-}$ is exactly L_{σ_-} and (Inc) implies that

$$m_t = \mu \left(\bigcup_{n \geq 1} L_{\sigma_n^-} \right) \leq \mu(L_{\sigma_-}) \leq \mu(L_{\sigma_*}) = m_t.$$

Thus, $m_t = \mu(L_{\sigma_-}) = \mu(L_{\sigma_*})$ and $\sigma_- = \sigma_*$ by (Inc) again.

Assume now that we are in Case (III) of Lemma 2.12 with $\sigma_- \in \text{Br}(T) \cup \{\rho\}$. We keep the same notation. We easily get

$$\bigcup_{n \geq 1} L_{\sigma_n^-} = \overline{\bigcup_{n \geq 1} L_{\sigma_n^-}} = L_{\sigma_-} \cup \left(\bigcup_{i \in K} C_{\sigma_-}(i) \right) \subset L_{\sigma_*}.$$

Suppose that $\sigma_- \neq \sigma_*$. Then $\sigma_- < \sigma_*$. It implies that σ_* is in $T \setminus \llbracket \rho, \sigma_- \rrbracket$. Let C be the connected component of $T \setminus \llbracket \rho, \sigma_- \rrbracket$ such that $\sigma_* \in C$. Clearly, we get

$$L_{\sigma_-} \cup \left(\bigcup_{i \in K} C_{\sigma_-}(i) \right) \leq C.$$

Observe that it is always possible to find $\sigma' \in C$ such that $\sigma' < \sigma_*$. So we get

$$\bigcup_{n \geq 1} L_{\sigma_n^-} = L_{\sigma_-} \cup \left(\bigcup_{i \in K} C_{\sigma_-}(i) \right) \subset L_{\sigma'} \subsetneq L_{\sigma_*}.$$

But (Inc) implies

$$m_t = \mu \left(\bigcup_{n \geq 1} L_{\sigma_n^-} \right) \leq \mu(L_{\sigma'}) < \mu(L_{\sigma_*}) = m_t,$$

which is impossible. Consequently, (11) holds and we also have proved that

$$L_{\sigma_-} \setminus \{\sigma_-\} \subset \bigcup_{n \geq 1} L_{\sigma_n^-} \subset L_{\sigma_-} = G_t. \quad (12)$$

Observe that $\sigma_* = \sigma_-$ does not depend on any sequence σ_n^- , $n \geq 1$ satisfying the assumptions of the lemma. Consequently, (12) remains true for any such sequence. We next claim that

$$\forall \sigma' \in T \setminus G_t, \quad \mu(L_{\sigma'}) > M_t. \quad (13)$$

Indeed, suppose that there is $\sigma' \in T \setminus G_t$ such that $\mu(L_{\sigma'}) = M_t$. It implies

$$M_t = \mu(L_{\sigma'}) > t \geq m_t = \mu(L_{\sigma_-}).$$

Thus, $\sigma_- < \sigma'$ by (Inc). But we can always find σ'' such that $\sigma_- < \sigma'' < \sigma'$. Since $G_t = L_{\sigma_-}$, it implies that $\sigma'' \in T \setminus G_t$ and then by (Inc)

$$M_t \leq \mu(L_{\sigma''}) < \mu(L_{\sigma'}) = M_t,$$

which is impossible. Therefore (13) holds.

Let σ_n^+ , $n \geq 1$, satisfying the assumptions of the lemma. By Lemma 2.13, it has a limit that we denote by σ_+ . Let σ_n^{i+} , $n \geq 1$, satisfying the same assumptions. (13) implies that for any $n \geq 1$,

$$\mu(L_{\sigma_n^+}) \text{ and } \mu(L_{\sigma_n^{i+}}) > M_t \quad \text{and} \quad \lim_{n \rightarrow \infty} \mu(L_{\sigma_n^+}) = \lim_{n \rightarrow \infty} \mu(L_{\sigma_n^{i+}}) = M_t.$$

So, by (Inc) we can construct a sequence σ_n^{i+} , $n \geq 1$, that also satisfies the assumptions of the lemma and such that it contains an infinite number of terms of the two sequences σ_n^+ , $n \geq 1$, and σ_n^{i+} , $n \geq 1$. Lemma 2.13 implies that σ_n^{i+} , $n \geq 1$, is convergent. Therefore the limit of σ_n^{i+} , $n \geq 1$ has to be also σ_+ . Thus, σ_+ does not depend on a choice of a sequence satisfying the assumptions of the lemma.

Let us fix such a sequence σ_n^+ , $n \geq 1$. We claim that

$$D_t = \bigcap_{n \geq 1} L_{\sigma_n^+}. \quad (14)$$

Indeed, observe first that

$$D_t \subset \bigcap_{n \geq 1} L_{\sigma_n^+}.$$

Suppose that there is $\sigma' \in \bigcap_{n \geq 1} L_{\sigma_n^+} \setminus D_t$. Then, we have $\mu(L_{\sigma_n^+}) \geq \mu(L_{\sigma'})$ for any $n \geq 1$. It implies that $M_t \geq \mu(L_{\sigma'})$, which actually implies $M_t > \mu(L_{\sigma'})$ by (13). Thus, σ' is in G_t . But, it implies

$$L_{\sigma'} \subset \bigcap_{T \setminus G_t} L_{\sigma} = D_t,$$

which rises a contradiction. Consequently (14) holds true.

To complete the proof of the lemma in the first case, it remains to prove that $\sigma_+ \in \llbracket \rho, \sigma_- \rrbracket$: we clearly have

$$L_{\sigma_+} \subset \bigcap_{n \geq 1} L_{\sigma_n^+} = D_t.$$

Thus, $\mu(L_{\sigma_+}) \leq M_t$, which actually implies $\mu(L_{\sigma_+}) < M_t$ by (13) and we get $\sigma_+ \in G_t = L_{\sigma_-}$. Observe now that $\sigma_+ \in \overline{T \setminus L_{\sigma_-}}$ as the limit of the σ_n^+ 's. Thus,

$$\sigma_+ \in L_{\sigma_-} \cap \overline{T \setminus L_{\sigma_-}} \subset \llbracket \rho, \sigma_- \rrbracket,$$

which completes the proof of the lemma in Case 1.

- *Case 2:* We now suppose

$$\forall \sigma' \in G_t, \quad \mu(L_{\sigma'}) < m_t. \quad (15)$$

By arguments similar to those used to prove uniqueness for σ_+ in Case 1, we prove that there exists $\sigma_- \in T$ such that any non-decreasing sequence σ_n^- , $n \geq 1$, that satisfies the assumptions of the lemma converges to σ_- .

Consider such a sequence σ_n^- , $n \geq 1$, and note that

$$\bigcup_{n \geq 1} L_{\sigma_n^-} \subset G_t.$$

Suppose there exists $\sigma' \in G_t \setminus \bigcup_{n \geq 1} L_{\sigma_n^-}$. Then for any $n \geq 1$, $\mu(L_{\sigma'}) \geq \mu(L_{\sigma_n^-})$ and we get $\mu(L_{\sigma'}) \geq m_t$, which contradicts (15). Consequently,

$$\bigcup_{n \geq 1} L_{\sigma_n^-} = G_t. \quad (16)$$

We distinguish two subcases whether it exists $\sigma_* \in T \setminus G_t$ such that $\mu(L_{\sigma_*}) = M_t$ or not.

– *Case 2.1:* Suppose there exists such a $\sigma_* \in T \setminus G_t$. Then σ_* is the minimal element of $T \setminus G_t$ and by (9) we get

$$D_t = L_{\sigma_*} \quad \text{and} \quad G_t = L_{\sigma_*} \setminus \{\sigma_*\} \quad (17)$$

and we get

$$\tilde{\sigma} = \sigma_* = \sigma_- . \quad (18)$$

Assume that there exists σ_n^+ , $n \geq 1$, a decreasing sequence satisfying assumptions of the lemma. It has a limit denoted by σ_+ . By previously used arguments, we can prove that any decreasing

sequence satisfying assumptions of the lemma converges to σ_+ . Recall that we have $D_t = L_{\sigma_*} = L_{\sigma_-}$. Suppose that there is $\sigma' \in \bigcap_{n \geq 1} L_{\sigma_n^+} \setminus D_t$. Then we get by (Inc)

$$M_t = \mu(L_{\sigma_-}) < \mu(L_{\sigma'}) \leq \mu \left(\bigcap_{n \geq 1} L_{\sigma_n^+} \right) = M_t,$$

which is absurd. It implies $D_t = L_{\sigma_*} = L_{\sigma_-} = \bigcap_{n \geq 0} L_{\sigma_n^+}$. By (17) and by the form of $\bigcap_{n \geq 1} L_{\sigma_n^+}$ given by Lemma 2.13, either $\sigma_- = \sigma_+$, either σ_- is in a connected component of $T \setminus \{\sigma_+\}$ that does not contain the root. Thus, it shows

$$\sigma_+ \in \llbracket \rho, \sigma_- \rrbracket ,$$

which completes the proof of the lemma in Case 2.1.

– *Case 2.2:* We suppose that (13) holds. By arguments similar to those used previously, we prove that there exists $\sigma_+ \in T$ such that any decreasing sequence σ_n^+ , $n \geq 1$, that satisfies the assumptions of the lemma converges to σ_+ . Consider such a sequence σ_n^+ , $n \geq 1$. We claim that

$$D_t = \bigcap_{n \geq 1} L_{\sigma_n^+} = G_t . \quad (19)$$

First note that $D_t \subset \bigcap_{n \geq 1} L_{\sigma_n^+}$. If $\sigma \in \bigcap_{n \geq 1} L_{\sigma_n^+}$, then $\mu(L_\sigma) \leq \mu(L_{\sigma_n^+})$ for any $n \geq 1$. So $\mu(L_\sigma) \leq M_t$ and thus, $\sigma \in G_t$ by (13). But clearly $G_t \subset D_t$, which completes the proof of (19).

It remains to prove

$$\sigma_+ \in \llbracket \rho, \sigma_- \rrbracket . \quad (20)$$

First, observe that by (19), $\sigma_+ \in G_t$. But it is the limit of the σ_n^+ 's that are in $T \setminus G_t$. Thus,

$$\sigma_+ \in \overline{T \setminus G_t} \cap G_t .$$

Recall (16). Then by Lemma 2.12. Deduce that

$$\overline{T \setminus G_t} \cap G_t \subset \llbracket \rho, \sigma_- \rrbracket ,$$

which implies (20) and the proof of the lemma is completed. ■

Definition 3.1 (*Exploration mapping associated with μ*) For any $t \in [0, M]$, we set $\phi(t) = \sigma_-$, where σ_- is defined by Lemma 3.1 and Remark 3.1.

Lemma 3.2 *The exploration mapping ϕ is left-continuous with right-limits. Moreover, $\phi(0+) = \rho$ and for any $t \in [0, M)$, $\phi(t+) \in \llbracket \rho, \phi(t) \rrbracket$.*

Proof: Fix $t \in (0, M]$. We first prove that ϕ is left-continuous at t . We define t_0 by

$$t_0 = \sup \{ \mu(L_\sigma) , \sigma \in T : \mu(L_\sigma) < t \} .$$

Recall that

$$m_t = \sup \{ \mu(L_\sigma) , \sigma \in T : \mu(L_\sigma) \leq t \} .$$

We distinguish three cases:

- *Case 1:* $t_0 < m_t$. Then, clearly $t = m_t = \mu(L_{\phi(t)})$ and $G_t = L_{\phi(t)}$. Next observe that for any $s \in [t_0, t)$, $G_s = L_{\phi(t)} \setminus \{\phi(t)\}$ and thus $\phi(s) = \phi(t)$, which implies that ϕ is left-continuous at t .

- *Case 2:* $t_0 = m_t < t$. If $s \in [m_t, t]$, then $m_s = m_t$ and $\phi(s) = \phi(t)$, which also implies that ϕ is left-continuous at t .

- *Case 3:* $t_0 = m_t = t$. Thus we can find an *increasing* sequence σ_n^- , $n \geq 1$, converging to $\sigma_- = \phi(t)$ and such that $\lim_{n \rightarrow \infty} \mu(L_{\sigma_n^-}) = t$. Let t_k , $k \geq 1$, be any increasing sequence of $[0, M]$ converging to t . We first claim that

$$\lim_{k \rightarrow \infty} m_{t_k} = t . \quad (21)$$

Clearly $m_{t_k} \leq t_k < t$. Set $t_n^- = \mu(L_{\sigma_n^-})$. By definition, $t_n^- = m_{t_n^-}$. Since $t_n^- < t$, then for any $n \geq 1$, we can find $k_n \geq 1$, such that $t_n^- \leq t_{k_n}$, which implies that $m_{t_n^-} = t_n^- \leq m_{t_{k_n}}$. Thus, (21) follows since $\lim_{n \rightarrow \infty} m_{t_n^-} = m_t = t$.

Without loss of generality we can assume that the sequence m_{t_k} , $k \geq 1$, is increasing. Use Lemma 3.1 at each t_k to find a sequence γ_k , $k \geq 1$, such that

$$d(\gamma_k, \phi(t_k)) \leq 2^{-k} \quad \text{and} \quad 0 \leq m_{t_k} - \mu(L_{\gamma_k}) < 2^{-k} \wedge (m_{t_k} - \mu(L_{\gamma_{k-1}})) \quad (22)$$

(observe that strict inequality is possible because $m_{t_{k-1}} < m_{t_k}$). Then, we get

$$\mu(L_{\gamma_{k-1}}) < \mu(L_{\gamma_k}) \leq m_t = t ,$$

which implies that γ_k , $k \geq 1$, is an increasing sequence of G_t . Moreover by (21), we get $\lim_{k \rightarrow \infty} \mu(L_{\gamma_k}) = t = m_t$. Lemma 3.1 implies that $\lim_{k \rightarrow \infty} d(\gamma_k, \phi(t)) = 0$ and, by definition of the sequence γ_k , $k \geq 1$, it implies that $\lim_{k \rightarrow \infty} d(\phi(t_k), \phi(t)) = 0$. This proves that ϕ is left-continuous at t in Case 3.

Existence of right-limits of ϕ at $t \in [0, M)$ is treated similarly: if $t < M_t$, then for any $s \in [t, M_t)$, we clearly have $m_s = m_t$, $M_s = M_t$ and $\phi(s) = \phi(t)$. Thus, ϕ has a right-limit at t , which is $\phi(t)$.

If we now assume that $M_t = t$, then we can find a *decreasing* sequence σ_n^+ , $n \geq 1$, converging to σ_+ and such that $\lim_{n \rightarrow \infty} \mu(L_{\sigma_n^+}) = M_t = t$. Let t_k , $k \geq 1$, be any decreasing sequence of $(0, M]$ converging to t . Set $t_n^+ = \mu(L_{\sigma_n^+})$, $n \geq 1$. We can find two increasing subsequences of indices $n(1, k)$, $n(2, k)$, $k \geq 1$ such that

$$t_{n(1, k+1)}^+ = \mu(L_{\sigma_{n(1, k+1)}^+}) < t_{n(2, k)} < t_{n(1, k)}^+ = \mu(L_{\sigma_{n(1, k)}^+}) .$$

Consequently

$$t_{n(1, k+1)}^+ \leq m_{t_{n(2, k)}} \leq M_{t_{n(2, k)}} \leq t_{n(1, k)}^+ . \quad (23)$$

It implies that m_{t_k} , $k \geq 1$ converges to t . Without loss of generality we can assume that m_{t_k} , $k \geq 1$ is a decreasing sequence. Use Lemma 3.1 at each t_k to find a sequence γ_k , $k \geq 1$ such that

$$d(\gamma_k, \phi(t_k)) \leq 2^{-k} \quad \text{and} \quad 0 \leq m_{t_k} - \mu(L_{\gamma_k}) < 2^{-k} \wedge (\mu(L_{\gamma_{k-1}}) - m_{t_k}) . \quad (24)$$

Then, by (23) and (24)

$$t < m_{t_{k+1}} < \mu(L_{\gamma_k}) < \mu(L_{\gamma_{k-1}}) \leq m_{t_{k-1}} ,$$

which implies that γ_k , $k \geq 1$ is a decreasing sequence of $T \setminus G_t$ such that $\lim_{k \rightarrow \infty} \mu(L_{\gamma_k}) = M_t = t$. Lemma 3.1 implies $\lim_{k \rightarrow \infty} d(\gamma_k, \sigma_+) = 0$. By definition of the sequence γ_k , $k \geq 1$, it implies that $\lim_{k \rightarrow \infty} d(\phi(t_k), \sigma_+) = 0$. This proves that ϕ has a right-limit at t and also that $\phi(t+) = \sigma_+$, where σ_+ is the point associated with t as defined in Lemma 3.1.

It remains to prove that ϕ is right-continuous at 0. If $\mu(\{\rho\}) > 0$, then $M_0 > 0 = m_0$ and we are in Case 1 or in Case 2. Assume that $\mu(\{\rho\}) = 0$. Fix a sequence t_k , $k \geq 1$ that decreases to 0. Let σ_n , $n \geq 1$, be a decreasing sequence of T such that $\{\rho\} = \bigcap_{n \geq 1} L_{\sigma_n}$. (Inc) implies that

$$\lim_{n \rightarrow \infty} \mu(L_{\sigma_n}) = \mu(\{\rho\}) = 0 .$$

Let $n \geq 1$. For all sufficiently large k we get $t_k < \mu(L_{\sigma_n})$, which implies that $\phi(t_k) \in D_{t_k} \subset L_{\sigma_n}$. Then any limit point γ of the sequence $\phi(t_k)$, $k \geq 1$, is in L_{σ_n} , for any $n \geq 1$. This implies that ρ is the only limit point of the sequence $\phi(t_k)$, $k \geq 1$ and the proof of the lemma is now completed. \blacksquare

Let us set

$$h(t) = d(\rho, \phi(t)) , \quad t \in [0, M].$$

Clearly h is left-continuous with right-limit; we also have $h(0) = h(0+) = 0$. Recall that $\phi(t) = \sigma_-$ and that if $\phi(t) \neq \phi(t+)$, then $\phi(t+) = \sigma_+$ with the notation of Lemma 3.1. Since $\sigma_+ \in \llbracket \rho, \sigma_- \rrbracket$, we get $h(t+) \leq h(t)$ (note that if $\phi(t) = \phi(t+)$, then $\phi(t+)$ is not necessarily equal to σ_+). Thus, h is in \mathcal{H}_M .

Proposition 3.3 *There exists an isometry j_h from (T_h, d_h) onto (T, d) such that $j(\rho_h) = \rho$ and such that*

$$\xi_1 \leq_h \xi_2 \implies j_h(\xi_1) \leq j_h(\xi_2) .$$

Proof: We first claim that for any $t_1 < t_2 < t_3$ in $[0, M]$,

$$d(\rho, \phi(t_2)) \geq d(\rho, \phi(t_1) \wedge \phi(t_3)) . \quad (25)$$

First observe that if $t_2 \in [m_{t_1}, M_{t_1})$, then clearly $\phi(t_2) = \phi(t_1)$. By left-continuity, we also get $\phi(M_{t_1}) = \phi(t_1)$. Similarly, if $t_2 \in [m_{t_3}, M_{t_3}]$, then $\phi(t_2) = \phi(t_3)$. Consequently, (25) holds for any $t_2 \in [m_{t_1}, M_{t_1}] \cup [m_{t_3}, M_{t_3}]$.

Let us assume that $M_{t_1} < t_2 < m_{t_3}$, which implies

$$m_{t_1} < m_{t_2} < m_{t_3} . \quad (26)$$

By Lemma 3.1, we can find three non-decreasing sequences $\sigma_n^-(i)$, $n \geq 1$, $i \in \{1, 2, 3\}$ such that

$$\sigma_n^-(i) \in G_{t_i}, \quad \lim_{n \rightarrow \infty} d(\sigma_n^-(i), \phi(t_i)) = 0, \quad \lim_{n \rightarrow \infty} \mu(L_{\sigma_n^-(i)}) = m_{t_i}.$$

Inequality (26) implies that for all sufficiently large n , $\mu(L_{\sigma_n^-(1)}) < \mu(L_{\sigma_n^-(2)}) < \mu(L_{\sigma_n^-(3)})$ and by (Inc)

$$\sigma_n^-(1) < \sigma_n^-(2) < \sigma_n^-(3). \quad (27)$$

Set $\sigma_0 = \phi(t_1) \wedge \phi(t_3)$ and let γ be such that

$$\llbracket \rho, \gamma \rrbracket = \llbracket \rho, \phi(t_2) \rrbracket \cap (\llbracket \rho, \phi(t_1) \rrbracket \cup \llbracket \rho, \phi(t_3) \rrbracket).$$

Suppose that $\gamma \in \llbracket \rho, \sigma_0 \rrbracket$. Then $\phi(t_2) \wedge \phi(t_3) = \phi(t_2) \wedge \phi(t_1) = \gamma$. Then $\phi(t_2) \wedge \sigma_0 = \gamma$. Now observe that for any $\sigma \in T$

$$2d(\sigma \wedge \sigma_0, \sigma_0) = d(\sigma_0, \sigma) + d(\rho, \sigma_0) - d(\sigma, \rho).$$

Thus, the application $\sigma \rightarrow d(\sigma \wedge \sigma_0, \sigma_0)$ is continuous. Consequently, for all sufficiently large n

$$d(\sigma_n^-(2) \wedge \sigma_0, \sigma_0) > \frac{2}{3}d(\gamma, \sigma_0) \quad \text{and} \quad d(\sigma_n^-(i) \wedge \sigma_0, \sigma_0) < \frac{1}{3}d(\gamma, \sigma_0), \quad i \in \{1, 3\}. \quad (28)$$

Let γ_n be such that

$$\llbracket \rho, \gamma_n \rrbracket = \llbracket \rho, \sigma_n^-(1) \rrbracket \cap (\llbracket \rho, \sigma_n^-(2) \rrbracket \cup \llbracket \rho, \sigma_n^-(3) \rrbracket).$$

(28) implies that for all sufficiently large n , the point γ_n is not in $\llbracket \rho, \sigma_n^-(2) \rrbracket$, which contradicts (27) by (Or2). Then, $\gamma \notin \llbracket \rho, \sigma_0 \rrbracket$ and γ is necessarily in $\llbracket \phi(t_1), \phi(t_3) \rrbracket$. Then, we get

$$d(\rho, \phi(t_2)) = d(\rho, \sigma_0) + d(\gamma, \sigma_0) + d(\gamma, \phi(t_2)),$$

which implies (25).

We keep notation $\sigma_0 = \phi(t_1) \wedge \phi(t_3)$ and we now prove that

$$\inf_{t \in [t_1, t_3]} d(\rho, \phi(t)) = d(\rho, \sigma_0). \quad (29)$$

To avoid triviality we suppose that $\sigma_0 \notin \{\phi(t_1), \phi(t_3)\}$. By Lemma 3.1 and the form of D_{t_1} and G_{t_3} given by Lemmas 2.12 and 2.13 we get $D_{t_1} \subset G_{t_3}$, which implies that $\phi(t_1) < \phi(t_3)$. Let σ_n , $n \geq 1$, be a sequence in $\llbracket \sigma_0, \phi(t_3) \rrbracket$ that decreases to σ_0 . By Lemma 3.1 and the form of D_{t_1} and G_{t_3} given by Lemma 2.12 and Lemma 2.13, we get for all $n \geq 1$,

$$D_{t_1} \subsetneq L_{\sigma_n} \subsetneq G_{t_3}.$$

Set $s_n = \mu(L_{\sigma_n})$, $n \geq 1$. The previous observation implies that for any $n \geq 1$,

$$t_1 \leq M_{t_1} \leq s_n \leq m_{t_3} \leq t_3.$$

Set $t = \mu(\bigcap_{n \geq 1} L_{\sigma_n}) = \lim_{n \rightarrow \infty} s_n$. Then $t \in [t_1, t_3]$. Next, by definition of ϕ , we have $\phi(s_n) = \sigma_n$. Since the sequence s_n , $n \geq 1$, decreases to t , we get $\phi(t+) = \sigma_0$. This, combined with (25) implies (29).

Now observe that (29) easily implies that for any $s, t \in [0, M]$,

$$d(\phi(s), \phi(t)) = h(s) + h(t) - 2 \inf_{u \in [s \wedge t, s \vee t]} h(u) = d_h(s, t) . \quad (30)$$

Recall that p_h stands for the canonical projection from $[0, M]$ to T_h . It implies that if $p_h(s) = p_h(t)$, then $\phi(s) = \phi(t)$. Thus, it makes sense to define $j_h : T_h \rightarrow T$ by $j_h(\xi) = \phi(s)$ for any $s \in p_h^{-1}(\{\xi\})$. Then (30) implies that j_h is an isometry from (T_h, d_h) onto (T, d) . Moreover we get $j_h(\rho_h) = \rho$.

It remains to prove that j_h is increasing: let $\sigma \in T \setminus \{\rho\}$. It is always possible to find an increasing sequence σ_n^- , $n \geq 1$, that converges to σ . Lemma 2.12 Case (I) implies that

$$\bigcup_{n \geq 1} L_{\sigma_n^-} = L_\sigma \setminus \{\sigma\}$$

and thus

$$\lim_{n \rightarrow \infty} \mu(L_{\sigma_n^-}) = \mu(L_\sigma \setminus \{\sigma\}) .$$

Lemma 3.1 implies that $\phi(\mu(L_\sigma \setminus \{\sigma\})) = \sigma$. Now observe that if $\phi(t) = \sigma$, then Lemma 3.1 easily implies that

$$L_\sigma \setminus \{\sigma\} \subset G_t .$$

Thus,

$$\mu(L_\sigma \setminus \{\sigma\}) \leq \mu(G_t) = m_t \leq t .$$

It proves that for any $\sigma \in T$

$$\inf \phi^{-1}(\{\sigma\}) = \mu(L_\sigma \setminus \{\sigma\}) . \quad (31)$$

Consequently, if $\sigma_1 < \sigma_2$, then

$$\inf \phi^{-1}(\{\sigma_1\}) = \mu(L_{\sigma_1} \setminus \{\sigma_1\}) \leq \mu(L_{\sigma_1}) \leq \mu(L_{\sigma_2} \setminus \{\sigma_2\}) = \inf \phi^{-1}(\{\sigma_2\}) .$$

Apply this inequality to $\sigma_1 = j_h(\xi_1)$ and $\sigma_2 = j_h(\xi_2)$ and observe that

$$\inf \phi^{-1}(\{\sigma_i\}) = \inf p_h^{-1}(\{\xi_i\}) , \quad i \in \{1, 2\}$$

to complete the proof of the proposition. ■

We next prove the following proposition.

Proposition 3.4 *We have $\mu = \mu_h \circ j_h^{-1}$. Furthermore the function h satisfies (Min).*

Proof: We first introduce some notation. Fix $\sigma \in T$ and recall notation \mathcal{C}_σ and \mathcal{C}_σ^+ from Section 2.3. By Lemmas 2.5 and 2.9, all the connected components in \mathcal{C}_σ and \mathcal{C}_σ^+ can be ordered by \leq . Define the following collection Σ_σ of families of connected components:

$$\Sigma_\sigma = \left\{ S \subset \mathcal{C}_\sigma \cup \mathcal{C}_\sigma^+ : \forall C \in S, \forall C' \in \mathcal{C}_\sigma \cup \mathcal{C}_\sigma^+ : (C' \leq C) \implies (C' \in S) \right\} .$$

We next define the two following sets of real numbers:

$$A_\sigma = \left\{ \mu(L_\sigma) + \sum_{C \in S} \mu(C) , S \in \Sigma_\sigma \right\} \quad \text{and} \quad B_\sigma = \left\{ \mu(L_\sigma) + \sum_{C \in S \cap \mathcal{C}_\sigma} \mu(C) , S \in \Sigma_\sigma \right\} .$$

We first prove the following lemma

Lemma 3.5 *For any $\sigma \in T$, one has*

$$F(h) \cap \phi^{-1}(\{\sigma\}) = [\mu(L_\sigma \setminus \{\sigma\}), \mu(L_\sigma)] , \quad (32)$$

$$S(h) \cap \phi^{-1}(\{\sigma\}) \subset B_\sigma \setminus \{\mu(L_\sigma)\} , \quad (33)$$

and

$$\phi^{-1}(\llbracket \rho, \sigma \rrbracket) \cap (\mu(L_\sigma), M] \subset (A_\sigma \setminus \{\mu(L_\sigma)\}) \cap S(h) . \quad (34)$$

Proof: For any $t \in [\mu(L_\sigma \setminus \{\sigma\}), \mu(L_\sigma)]$, we get $m_t = \mu(L_\sigma \setminus \{\sigma\})$ and $M_t = \mu(L_\sigma)$; the definition of ϕ and (31) imply that

$$[\mu(L_\sigma \setminus \{\sigma\}), \mu(L_\sigma)] \subset F(h) \cap \phi^{-1}(\{\sigma\}) . \quad (35)$$

Let $t \in \phi^{-1}(\{\sigma\}) \cap (\mu(L_\sigma), M]$. Thus

$$\mu(L_\sigma) \leq m_t \leq t \leq M_t . \quad (36)$$

Let σ_n^- , $n \geq 1$ be a non-decreasing sequence such that

$$\lim_{n \rightarrow \infty} \mu(L_{\sigma_n^-}) = m_t .$$

Then Lemma 3.1 implies that $\lim_{n \rightarrow \infty} d(\sigma_n^-, \sigma) = 0$ and

$$D_t \setminus \{\sigma\} = G_t \setminus \{\sigma\} \subset \bigcup_{n \geq 1} L_{\sigma_n^-} \subset G_t \subset D_t . \quad (37)$$

Suppose that the sequence σ_n^- , $n \geq 1$ corresponds to Case (I) or Case (II) in Lemma 2.12. Then (37) implies

$$G_t \setminus \{\sigma\} = D_t \setminus \{\sigma\} = L_\sigma \setminus \{\sigma\} .$$

It implies that $M_t = \mu(D_t) \leq \mu(L_\sigma)$, which contradicts (36). Then, σ_n^- , $n \geq 1$ corresponds to Case (III) in Lemma 2.12. Consequently

$$D_t = G_t = L_\sigma \cup \left(\bigcup_{k \in K} C_\sigma(k) \right) , \quad (38)$$

with the same definition of $K \subset I_\sigma$ as in Lemma 2.12. Thus it implies

$$m_t = \mu(G_t) = \mu(D_t) = M_t = t .$$

Set $S = \{C_\sigma(k) , k \in K\}$. The definition of K in Lemma 2.12 implies that $S \in \Sigma_\sigma$. Thus,

$$t = \mu(G_t) = \mu(L_\sigma) + \sum_{k \in K} \mu(C_\sigma(k)) \in B_\sigma \setminus \{\mu(L_\sigma)\} .$$

Let $k_0 \in K$ and $\sigma' \in C_\sigma(k_0)$. Observe that

$$L_\sigma \subset L_{\sigma'} \subset G_t .$$

Set $t' = \mu(L_{\sigma'})$. Then $\mu(L_{\sigma}) < t' \leq t$ and $\phi(t') = \sigma' \neq \sigma$. It implies that $t \in S(h)$. It completes the proof of (32) and (33).

Let us prove (34). First note that

$$\phi^{-1}(\llbracket \rho, \sigma \rrbracket) \cap (\mu(L_{\sigma}), M] \subset S(h)$$

and (33) implies that $\phi^{-1}(\{\sigma\}) \cap (\mu(L_{\sigma}), M] \subset S(h)$. Thus

$$\phi^{-1}(\llbracket \rho, \sigma \rrbracket) \cap (\mu(L_{\sigma}), M] \subset S(h).$$

Let $t \in (\mu(L_{\sigma}), M]$ be such that $j_h(t) = \sigma' \in \llbracket \rho, \sigma \rrbracket$. Then $t \in \phi^{-1}(\{\sigma'\}) \cap S(h)$. By (33), it implies that $t \in B_{\sigma'} \setminus \{\mu(L_{\sigma'})\}$. Then there exists $S' \in \Sigma_{\sigma'}$ such that

$$t = \mu(L_{\sigma'}) + \sum_{C' \in S'} \mu(C'). \quad (39)$$

Since $\sigma' \in \llbracket \rho, \sigma \rrbracket$, there is $C_0 \in S'$ such that $\sigma \in C_0$. Recall from Section 2.3 notation C_j^+ and γ_j^+ , $j \in J_{\sigma}^+$ and set

$$A = \{C_j^+, j \in J_{\sigma}^+ : d(\sigma, \gamma_j^+) < d(\sigma, \sigma')\}$$

and

$$B = \{C_j^+, j \in J_{\sigma}^+ : d(\sigma, \gamma_j^+) = d(\sigma, \sigma') \text{ and } C_j^+ \in S'\}.$$

Observe that $C_{\sigma} \cup A \cup B \in \Sigma_{\sigma}$ and that

$$L_{\sigma'} \cup \left(\bigcup_{C' \in S'} C' \right) = L_{\sigma} \cup \left(\bigcup_{k \in I_{\sigma}} C_{\sigma}(k) \right) \cup \left(\bigcup_{C'' \in A \cup B} C'' \right).$$

It implies that $t \in A_{\sigma} \setminus \{\mu(L_{\sigma})\}$ and it completes the proof of the lemma. ■

Observe that by (31), we get $\phi(t) \leq \sigma$ for any $t \in [0, \mu(L_{\sigma} \setminus \{\sigma\})]$. If $\sigma' \in C$ with $C \in \mathcal{C}_{\sigma}^-$, then we get

$$L_{\sigma'} \cup \left(\bigcup_{C' \in \mathcal{C}_{\sigma'}} C' \right) \subset L_{\sigma} \setminus \{\sigma\}. \quad (40)$$

Thus, by (33) applied to σ' we get

$$\sup \phi^{-1}(\{\sigma'\}) \leq \mu(L_{\sigma'}) + \sum_{C' \in \mathcal{C}_{\sigma'}} \mu(C')$$

and (40) implies

$$\sup \phi^{-1}(\{\sigma'\}) \leq \mu(L_{\sigma} \setminus \{\sigma\}).$$

Consequently,

$$\phi^{-1}(L_{\sigma} \setminus \llbracket \rho, \sigma \rrbracket) = \bigcup_{C \in \mathcal{C}_{\sigma}^-} \phi^{-1}(C) \subset [0, \mu(L_{\sigma} \setminus \{\sigma\})].$$

This, combined with (34), implies that

$$[0, \mu(L_\sigma)] \subset \phi^{-1}(L_\sigma) \subset [0, \mu(L_\sigma)] \cup A_\sigma . \quad (41)$$

We need the following lemma.

Lemma 3.6 *For any $\sigma \in T$, A_σ is a Lebesgue null set.*

Proof: First observe that

$$\mu(L_\sigma) = \inf A_\sigma \leq \sup A_\sigma = \mu(L_\sigma) + \sum_{C \in \mathcal{C}_\sigma \cup \mathcal{C}_\sigma^+} \mu(C) = \mu(T) = M .$$

Set for any $C \in \mathcal{C}_\sigma \cup \mathcal{C}_\sigma^+$, $S(C) = \{C' \in \mathcal{C}_\sigma \cup \mathcal{C}_\sigma^+ : C' < C\}$. Clearly $S(C)$ and $S(C) \cup \{C\}$ are in Σ_σ . Thus, the real numbers $a(C)$ and $b(C)$ given by

$$a(C) = \mu(L_\sigma) + \sum_{C' \in S(C)} \mu(C') \quad \text{and} \quad b(C) = \mu(C) + a(C)$$

are in A_σ . Observe now that $\overline{A}_\sigma \cap (a(C), b(C)) = \emptyset$, where \overline{A}_σ stands for the closure of the set A_σ . Thus

$$\bigcup_{C \in \mathcal{C}_\sigma \cup \mathcal{C}_\sigma^+} (a(C), b(C)) \subset [\mu(L_\sigma), \mu(T)] \setminus \overline{A}_\sigma . \quad (42)$$

Note that

$$\lambda \left(\bigcup_{C \in \mathcal{C}_\sigma \cup \mathcal{C}_\sigma^+} (a(C), b(C)) \right) = \sum_{C \in \mathcal{C}_\sigma \cup \mathcal{C}_\sigma^+} \mu(C) = \mu(T) - \mu(L_\sigma) .$$

Thus, (42) implies that $\lambda(\overline{A}_\sigma) = 0$, which completes the proof of the lemma. ■

The previous lemma and (41) imply that for any $\sigma \in T$,

$$\lambda(\phi^{-1}(L_\sigma)) = \mu(L_\sigma) .$$

Thus, $\mu = \lambda \circ \phi^{-1}$ by Lemma 2.14. Consequently $\mu = \mu_h \circ j_h^{-1}$, by definition of j_h .

Lemma 3.7 *For any $h \in \mathcal{H}$, the set of times of first visit $F(h)$ and the set of times of latter visit $S(h)$ are Borel sets of the line.*

Proof: As already noticed, we have $p_h^{-1}(\text{Lf}(T_h)) \subset F(h)$. Thus,

$$S(h) \subset p_h^{-1}(\text{Sk}(T_h)) . \quad (43)$$

Let $t \in [0, \zeta(h)]$. We set

$$E_t = \left\{ s \in [t, \zeta(h)] : h(s) = \inf_{t \leq u \leq s} h(u) \quad \text{and} \quad h(s) < h(t) \right\} .$$

Clearly, E_t is a (possibly empty) Borel set of the line. Suppose that $E_t \neq \emptyset$. Let $s \in E_t$. Set $\ell = \sup\{u \leq t : h(u) \leq h(s)\}$. Then

$$h(\ell) = h(s) = \inf_{\ell \leq u \leq s} h(u) ,$$

that is $p_h(\ell) = p_h(s)$. Now observe that since $h(t) > h(s)$, $p_h(t) \neq p_h(s)$. Thus, $s \in S(h)$ since $\ell < t \leq s$. So we get

$$E_t \subset S(h) .$$

Let t_n , $n \geq 1$, be a sequence that is dense in $[0, \zeta(h)]$ and let $s \in S(h)$. Then, there exists $s' < s$ such that $p_h(s') = p_h(s)$. Since $p_h(s)$ is not a leaf, there exists $t_n \in (s', s)$ such that $h(t_n) > h(s)$. Then

$$h(s) = \inf_{t_n \leq u \leq s} h(u) < h(t_n) ,$$

which implies that $s \in E_{t_n}$. We thus have proved that

$$S(h) = \bigcup_{n \geq 1} E_{t_n} , \tag{44}$$

which implies the lemma. ■

It remains to prove that h satisfies (Min). Observe that for any $t \in [0, \zeta(h)]$, one has

$$E_t \subset \phi^{-1}([\rho, p_h(t)]) \cap (\mu(L_{p_h(t)}), M] \subset A_{p_h(t)} .$$

Then (44) implies

$$S(h) \subset \bigcup_{n \geq 1} A_{p_h(t_n)} ,$$

which implies (Min) by Lemma 3.6. This completes the proof of the proposition. ■

The following proposition completes the proof of Theorem 1.1.

Proposition 3.8 *Let h_1, h_2 be two functions in \mathcal{H} that satisfy (Min) and such that the two structured trees*

$$(T_{h_1}, d_{h_1}, \rho_{h_1}, \leq_{h_1}, \mu_{h_1}) \quad \text{and} \quad (T_{h_2}, d_{h_2}, \rho_{h_2}, \leq_{h_2}, \mu_{h_2})$$

are equivalent. Then, $h_1 = h_2$.

Proof: To simplify notation we assume that

$$(T_{h_1}, d_{h_1}, \rho_{h_1}, \leq_{h_1}, \mu_{h_1}) = (T_{h_2}, d_{h_2}, \rho_{h_2}, \leq_{h_2}, \mu_{h_2}) = (T, d, \rho, \leq, \mu) .$$

First observe that $\zeta(h_1) = \zeta(h_2) = \mu(T) = M$. Set for any $\sigma \in T$ and for $i \in \{1, 2\}$

$$\ell_i(\sigma) = \inf p_{h_i}^{-1}(\{\sigma\}) \quad \text{and} \quad r_i(\sigma) = \inf\{t > \ell_i(\sigma) : p_{h_i}(t) \neq \sigma\}$$

and recall that by Lemma 2.2, $p_{h_i}(\ell_i(\sigma)) = \sigma$. By definition, if $\sigma' < \sigma$, then $\ell_i(\sigma') < \ell_i(\sigma)$, for $i \in \{1, 2\}$. Thus

$$p_{h_i}([0, \ell_i(\sigma))) = L_\sigma \setminus \{\sigma\} , \quad \sigma \in T , \quad i \in \{1, 2\} . \tag{45}$$

Observe that

$$p_{h_i}([\ell_i(\sigma), r_i(\sigma)]) = \{\sigma\}, \quad \sigma \in T, \quad i \in \{1, 2\}. \quad (46)$$

Let $t \in (r_i(\sigma), M]$ be such that $p_{h_i}(t) \in L_\sigma$; observe that t is necessarily a time of latter visit. Thus,

$$p_{h_i}^{-1}(L_\sigma) \cap (r_i(\sigma), M] \subset S(h_i), \quad \sigma \in T, \quad i \in \{1, 2\}. \quad (47)$$

Then (45), (46) and (47) imply for $i \in \{1, 2\}$

$$[0, \ell_i(\sigma)) \subset p_{h_i}^{-1}(L_\sigma \setminus \{\sigma\}) \subset [0, \ell_i(\sigma)) \cup S(h_i) \quad \text{and} \quad [0, r_i(\sigma)] \subset p_{h_i}^{-1}(L_\sigma) \subset [0, r_i(\sigma)] \cup S(h_i).$$

Consequently,

$$\mu(L_\sigma \setminus \{\sigma\}) = \lambda(p_{h_i}^{-1}(L_\sigma \setminus \{\sigma\})) = \ell_i(\sigma) \quad \text{and} \quad \mu(L_\sigma) = \lambda(p_{h_i}^{-1}(L_\sigma)) = r_i(\sigma),$$

since h_1 and h_2 satisfy (Min). (46) then implies that h_1 and h_2 coincide on the set

$$F(h_1) = F(h_2) = \bigcup_{\sigma \in T} [\mu(L_\sigma \setminus \{\sigma\}), \mu(L_\sigma)]$$

that is a set of full Lebesgue measure in $[0, M]$ and $h_1 = h_2$ follows since h_1 and h_2 are left-continuous. ■

We now prove Theorem 1.3.

Proof of Theorem 1.3: Recall that j_h (resp. $j_{h'}$) stands for the isometry that maps the structured tree $(T_h, d_h, \rho_h, \leq_h, \mu_h)$ (resp. $(T_{h'}, d_{h'}, \rho_{h'}, \leq_{h'}, \mu_{h'})$) onto (T, d, ρ, \leq, μ) (resp. onto (T, d, ρ, \leq, μ')). Obviously, $\mu(T) = \zeta(h)$ and $\mu'(T) = \zeta(h')$. Let us denote by ϕ (resp. ϕ') the exploration mapping from $[0, \mu(T)]$ (resp. $[0, \mu'(T)]$) onto T associated with μ (resp. μ') as in Definition 3.1. Recall that Theorem 1.1 implies that

$$j_h \circ p_h = \phi \quad \text{and} \quad j_{h'} \circ p_{h'} = \phi'. \quad (48)$$

Let us first prove the existence of the time-change. Recall that (32) implies

$$F(h) = \bigcup_{\sigma \in T} [\mu(L_\sigma \setminus \{\sigma\}), \mu(L_\sigma)] \quad \text{and} \quad F(h') = \bigcup_{\sigma \in T} [\mu'(L_\sigma \setminus \{\sigma\}), \mu'(L_\sigma)]. \quad (49)$$

Since h and h' satisfy (Min), $F(h)$ and $F(h')$ are sets of full Lebesgue measure. Thus, they are dense in resp. $[0, \zeta(h)]$ and $[0, \zeta(h')]$. Observe that it is possible to find a non-negative application $\tilde{\varphi}$ on $F(h)$ that is non-decreasing and such that for any $\sigma \in T$

- (a) $\tilde{\varphi}(\mu(L_\sigma \setminus \{\sigma\})) = \mu'(L_\sigma \setminus \{\sigma\})$;
- (b) if $\mu(\{\sigma\}) > 0$, then $\tilde{\varphi}$ is left-continuous on $(\mu(L_\sigma \setminus \{\sigma\}), \mu(L_\sigma)]$ and $\tilde{\varphi}(\mu(L_\sigma)) \leq \mu'(L_\sigma)$.

Remark 3.2 If $\mu(\{\sigma\})\mu'(\{\sigma\}) > 0$, then observe that we can find infinitely many $\tilde{\varphi}$ satisfying (a) and (b). □

We define φ by

$$\varphi(t) = \sup\{\tilde{\varphi}(s) , s \in F(h) \text{ and } s \leq t\} , \quad t \in [0, \mu(T)] .$$

Observe that φ and $\tilde{\varphi}$ coincide on $F(h)$. Consequently $\varphi(0) = \tilde{\varphi}(0) = 0$.

Let $t \in (0, \mu(T)]$. We claim that there exists an increasing sequence $s_n \in [0, \mu(T)]$, $n \geq 1$, converging to t and such that

$$\lim_{n \rightarrow \infty} \varphi(s_n) = \varphi(t) . \quad (50)$$

Since φ is non-decreasing, the previous claim easily implies that φ is left-continuous at t . Let us prove (50): the result is clear if $t \notin F(h)$; the only non-trivial case to consider then, is when $t = \mu(L_\sigma \setminus \{\sigma\})$, $\sigma \in T \setminus \{\rho\}$. It is always possible to find an increasing sequence $\sigma_n \in \llbracket \rho, \sigma \rrbracket$, $n \geq 1$, that converges to σ and such that $\mu(\{\sigma_n\}) = 0$, $n \geq 1$. Thus, Lemma 2.12 implies

$$\bigcup_{n \geq 1} L_{\sigma_n} \setminus \{\sigma_n\} = \bigcup_{n \geq 1} L_{\sigma_n} = L_\sigma \setminus \{\sigma\} . \quad (51)$$

Set for any $n \geq 1$, $s_n = \mu(L_{\sigma_n} \setminus \{\sigma_n\})$. Clearly, $\varphi(s_n) = \tilde{\varphi}(s_n) = \mu'(L_{\sigma_n} \setminus \{\sigma_n\})$ and (51) implies

$$\lim_{n \rightarrow \infty} \varphi(s_n) = \lim_{n \rightarrow \infty} \mu'(L_{\sigma_n} \setminus \{\sigma_n\}) = \mu'(L_\sigma \setminus \{\sigma\}) = \varphi(t)$$

which implies (50).

Thus, we have constructed a non-decreasing, left-continuous mapping $\varphi : [0, \zeta(h)] \rightarrow [0, \infty)$ that coincides with $\tilde{\varphi}$ on $F(h)$ and such that $\varphi(0) = 0$. Moreover, (a) and (b) imply that $\phi = \phi' \circ \varphi$ on $F(h)$ by (49). It easily implies $h = h' \circ \varphi$ on $[0, \zeta(h)]$ since h and $h' \circ \varphi$ are left-continuous and since $F(h)$ is dense in $[0, \zeta(h)]$.

Let us prove the uniqueness result and the other points of Theorem 1.3. To that end, we need the following proposition.

Proposition 3.9 *Let (T, d, ρ) be a rooted compact real tree and let \leq be a linear order satisfying (Or1) and (Or2). Let μ and μ' be two finite Borel measures on T that both satisfy (Mes). Denote by h and h' the height functions associated with resp. (T, d, ρ, \leq, μ) and (T, d, ρ, \leq, μ') by Theorem 1.1 (h and h' then satisfy (Min)). Assume that $\varphi : [0, \zeta(h)] \rightarrow [0, \infty)$ is non-decreasing, left-continuous and such that*

$$\varphi(0) = 0 \quad \text{and} \quad h = h' \circ \varphi .$$

Then for any $\sigma \in T$, we get

$$\varphi(\mu(L_\sigma \setminus \{\sigma\})) = \mu'(L_\sigma \setminus \{\sigma\}) \quad \text{and} \quad \varphi(\mu(L_\sigma)) \leq \mu'(L_\sigma) .$$

Proof of Proposition 3.9: To simplify notation, we set $f = j_h^{-1} \circ j_{h'}$; then f maps the rooted ordered compact real tree $(T_{h'}, d_{h'}, \rho_{h'}, \leq_{h'})$ onto $(T_h, d_h, \rho_h, \leq_h)$. We first want to prove

$$j_h \circ p_h = j_{h'} \circ p_{h'} \circ \varphi . \quad (52)$$

First, let us fix $s_1, s_2 \in [0, \zeta(h)]$ and let us set $\sigma_1 = p_h(s_1)$ and $\sigma_2 = p_h(s_2)$. For any $\sigma \in T_h$, denote by $t_\sigma \in [0, \zeta(h)]$ a time such that $p_h(t_\sigma) = \sigma$; assume that $t_{\sigma_1} = s_1$ and $t_{\sigma_2} = s_2$. We then define G from T_h to T_h by

$$G(\sigma) = f(p_{h'}(\varphi(t_\sigma))) , \quad \sigma \in T_h . \quad (53)$$

We first get for any $\sigma, \sigma' \in T_h$

$$d_h(G(\sigma), G(\sigma')) = d_{h'}(p_{h'}(\varphi(t_\sigma)), p_{h'}(\varphi(t_{\sigma'}))) = d_{h'}(\varphi(t_\sigma), \varphi(t_{\sigma'}))$$

for f is an isometry. Then, note that

$$d_{h'}(\varphi(t_\sigma), \varphi(t_{\sigma'})) \geq d_h(t_\sigma, t_{\sigma'}) = d_h(\sigma, \sigma')$$

since $h = h' \circ \varphi$. Thus, for any $\sigma, \sigma' \in T_h$

$$d_h(G(\sigma), G(\sigma')) \geq d_h(\sigma, \sigma') .$$

Since the metric space (T_h, d_h) is compact, standard arguments imply that G is actually a bijective isometry (see Theorem 1.6.15 (2) in [9]). It first implies that $p_{h'} \circ \varphi$ is surjective:

$$p_{h'} \circ \varphi ([0, \zeta(h)]) = T_{h'} . \quad (54)$$

It also implies that $d_h(G(\sigma_1), G(\sigma_2)) = d_h(\sigma_1, \sigma_2)$. So, we have proved

$$d_h(s_1, s_2) = d_{h'}(\varphi(s_1), \varphi(s_2)) , \quad s_1, s_2 \in [0, \zeta(h)] . \quad (55)$$

Let us prove that G preserves \leq_h . To that end, we need to prove the following lemma.

Lemma 3.10 *Let $h, h' \in \mathcal{H}$ and $\varphi : [0, \zeta(h)] \rightarrow [0, \infty)$ be as in Proposition 3.9. Let $t \in [0, \zeta(h)]$ be such that $\varphi(t) < \varphi(t+)$. Then*

$$h'(u) = h'(\varphi(t)) = h(t) , \quad u \in [\varphi(t), \varphi(t+)] .$$

Proof of Lemma 3.10: We introduce

$$s_0 = \inf\{s \in [\varphi(t), \zeta(h')] : h'(s) \neq h'(\varphi(t))\} ,$$

with the convention that $\inf \emptyset = \infty$. Suppose that $s_0 < \varphi(t+)$. Then, $F(h') \cap (s_0, \varphi(t+))$ is non-empty for $F(h')$ is dense in $[0, \zeta(h')]$ by (Min). Consequently, we can find $s \in F(h') \cap (s_0, \varphi(t+))$ such that $h'(s) \neq h'(\varphi(t))$. There exists $u \in [0, \zeta(h)]$ such that $p_{h'}(\varphi(u)) = p_{h'}(s)$ since $p_{h'} \circ \varphi$ is surjective. Set $\sigma = p_{h'}(s)$. If $\varphi(u) < s$, then $\varphi(u) \leq \varphi(t)$ and since $s \in F(h')$, it implies

$$[\varphi(u), s] \subset [\mu'(L_\sigma \setminus \{\sigma\}), \mu'(L_\sigma)]$$

by (49). Consequently, we get $\sigma = p_{h'}(r)$, $r \in [\varphi(u), s]$. But $s_0 \in [\varphi(u), s]$, which rises a contradiction. Thus, $\varphi(u) > s$. It implies that $u > t$ and $\varphi(u) > \varphi(t+)$. Since $d_{h'}(\varphi(u), s) = 0$, we get

$$h'(\varphi(u)) = h'(s) = \inf_{r \in [s, \varphi(u)]} h'(r) \leq h'(\varphi(t+)) = h(t+) . \quad (56)$$

Now deduce from (55) that for any $\epsilon > 0$

$$d_h(t, t + \epsilon) = d_{h'}(\varphi(t), \varphi(t + \epsilon)) .$$

Observe that

$$\lim_{\epsilon \rightarrow 0} d_h(t, t + \epsilon) = h(t) - h(t+)$$

and that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} d_{h'}(\varphi(t), \varphi(t + \epsilon)) &= h'(\varphi(t)) + h'(\varphi(t+)) - 2 \left(h'(\varphi(t+)) \wedge \inf_{r \in [\varphi(t), \varphi(t+)]} h'(r) \right) \\ &= h(t) + h(t+) - 2 \left(h(t+) \wedge \inf_{r \in [\varphi(t), \varphi(t+)]} h'(r) \right) . \end{aligned}$$

Consequently,

$$h(t+) \leq \inf_{r \in [\varphi(t), \varphi(t+)]} h'(r) . \quad (57)$$

Then (56) and (57) both imply that $h'(s) = h(t+)$ for $\inf_{r \in [s, \varphi(u)]} h'(r) \leq h'(s)$. Since we have supposed that $h'(s) \neq h'(\varphi(t))$, we get

$$h'(s) = h(t+) < h(t) = h'(\varphi(t)) .$$

Now set

$$u_0 = \sup\{u \in [0, \varphi(t)) : h'(u) \leq h'(s) = h(t+)\} .$$

Clearly,

$$h'(u_0) = \inf_{r \in [u_0, s]} h'(r) = h'(s) = h(t+) .$$

Thus, $d_{h'}(u_0, s) = 0$. Since $s \in F(h')$, we get $[u_0, s] \subset [\mu'(L_\sigma \setminus \{\sigma\}), \mu'(L_\sigma)]$. Thus, for any $r \in [u_0, s]$, we get $p_{h'}(r) = \sigma$. But $s_0 \in [u_0, s]$ and

$$h'(s_0) = h'(\varphi(t)) > d(\rho, \sigma) = h'(s) .$$

Thus, $p_{h'}(s_0) \neq \sigma$, which rises a contradiction. It then proves that $s_0 \geq \varphi(t+)$, which implies the lemma. \blacksquare

Let us complete the proof of the proposition. Let $t \in (0, \zeta(h)]$ such that for any $s \in [0, t)$, $p_h(s) \neq p_h(t)$. Then, the previous lemma implies that

$$p_{h'}(u) \neq p_{h'}(\varphi(t)) , \quad u \in [0, \varphi(t)) . \quad (58)$$

(Suppose indeed that there exists $u \in [0, \varphi(t))$ such that $p_{h'}(u) = p_{h'}(\varphi(t))$ which is equivalent to $d_{h'}(u, \varphi(t)) = 0$; there exists $s \in [0, t)$ such that $\varphi(s) \leq u \leq \varphi(s+)$; the previous lemma implies that $d_{h'}(\varphi(s), u) = 0$; thus $d_{h'}(\varphi(s), \varphi(t)) = 0$; (55) implies $d_h(s, t) = 0$, which rises a contradiction.)

Consequently, we get for any $s_1, s_2 \in [0, \zeta(h)]$,

$$p_h(s_1) \leq_h p_h(s_2) \implies p_{h'}(\varphi(s_1)) \leq_{h'} p_{h'}(\varphi(s_2)) . \quad (59)$$

Thus, by definition of G

$$\sigma_1 \leq_h \sigma_2 \implies G(\sigma_1) \leq_h G(\sigma_2) . \quad (60)$$

Let us prove that (60) implies (52): Let us suppose that there exists $\sigma \in T_h$ such that $G(\sigma) \neq \sigma$. Then, either $G(\sigma) <_h \sigma$, either $\sigma <_h G(\sigma)$. Define recursively the sequence $G^n(\sigma)$, $n \geq 1$, by $G^{n+1}(\sigma) = G(G^n(\sigma))$ and $G^1(\sigma) = G(\sigma)$. In both cases the sequence $G^n(\sigma)$, $n \geq 1$, is (\leq_h) -monotone and thus convergent, by Proposition 2.11. Now, observe that $d_h(G(\sigma), \sigma) = d_h(G^{n+1}(\sigma), G^n(\sigma))$, which contradicts $G(\sigma) \neq \sigma$.

So, we have proved that $G(\sigma) = f(p_{h'}(\varphi(t_\sigma))) = \sigma$. By (55), this identity does not depend on the choice of t_σ in $p_h^{-1}(\{\sigma\})$. Consequently

$$f(p_{h'}(\varphi(t))) = p_h(t) , \quad t \in [0, \zeta(h)] ,$$

which implies (52).

Let us now complete the proof of Proposition 3.9: recall from (31) that for any $\sigma \in T$

$$\mu(L_\sigma \setminus \{\sigma\}) = \inf\{t \geq 0 : j_h(p_h(t)) = \sigma\} .$$

Then, (58) and (52) imply

$$\varphi(\mu(L_\sigma \setminus \{\sigma\})) = \inf\{t \geq 0 : j_{h'}(p_{h'}(t)) = \sigma\} = \mu'(L_\sigma \setminus \{\sigma\}) . \quad (61)$$

This proves the first point of Proposition 3.9.

Let us prove the last one. Fix $t \in [\mu(L_\sigma \setminus \{\sigma\}), \mu(L_\sigma)]$. (52) implies that

$$\sigma = j_{h'}(p_{h'}(\varphi(t))) = j_h(p_h(t)) , \quad t \in [\mu(L_\sigma \setminus \{\sigma\}), \mu(L_\sigma)] .$$

Suppose that $\varphi(t) \notin F(h')$. Thus, $\varphi(t) > \mu'(L_\sigma)$ by (49). Since $p_{h'}(\mu'(L_\sigma)) = p_{h'}(\varphi(t)) = \sigma$, there exists $s \in (\mu'(L_\sigma), \varphi(t))$ such that $p_{h'}(s) \neq \sigma$. Set $p_{h'}(s) = \sigma'$. It is easy to check that

$$p_{h'}^{-1}(\{\sigma'\}) \subset (\mu'(L_\sigma), \varphi(t)) .$$

Since $p_{h'} \circ \varphi$ is surjective, we can find $u < t$ such that

$$\varphi(u) \in (\mu'(L_\sigma), \varphi(t)) \quad \text{and} \quad \sigma' = j_{h'}(p_{h'}(\varphi(u))) . \quad (62)$$

It implies

$$\varphi(\mu(L_\sigma \setminus \{\sigma\})) = \mu'(L_\sigma \setminus \{\sigma\}) \leq \mu'(L_\sigma) < \varphi(u) \leq \varphi(t) .$$

Thus, $u \in [\mu(L_\sigma \setminus \{\sigma\}), t]$. (32) then implies that

$$\sigma = j_h(p_h(u)) = j_{h'}(p_{h'}(\varphi(u))) ,$$

which contradicts (62). Thus, it proves that $\varphi(t) \in F(h')$ and (49) implies that $\varphi(t) \leq \mu'(L_\sigma)$, which completes the proof of the proposition. \blacksquare

Let us complete the proof of Theorem 1.3: Observe that (iv) is Lemma 3.10. Let us prove (ii): if μ' has no atom, then Proposition 3.9 implies

$$\varphi(F(h)) = F(h') = \{\mu'(L_\sigma) : \sigma \in T\},$$

which implies that φ is continuous since $F(h)$ and $F(h')$ are dense.

Let us prove (iii): assume that μ has no atom. let $t_1 < t_2$ be in $(0, \mu(T))$; there exist $\sigma_1 < \sigma_2$ in T such that

$$t_1 < \mu(L_{\sigma_1}) < \mu(L_{\sigma_2}) < t_2,$$

for $F(h) = \{\mu(L_\sigma); \sigma \in T\}$ is dense in $[0, \mu(T)]$; it implies by (Inc) that

$$\varphi(t_1) \leq \varphi(\mu(L_{\sigma_1})) = \mu'(L_{\sigma_1} \setminus \{\sigma_1\}) < \mu'(L_{\sigma_2} \setminus \{\sigma_2\}) = \varphi(\mu(L_{\sigma_2})) \leq \varphi(t_2)$$

and thus $\varphi(t_1) < \varphi(t_2)$, which completes the proof of (iii).

Finally, let us prove (i). Assume that μ and μ' do not share any atom. Thus, Proposition 3.9 and (49) imply that φ is uniquely determined on $\{0\} \cup F(h)$. Consequently, φ is uniquely determined on $[0, \zeta(h)]$ for $F(h)$ is dense in $[0, \zeta(h)]$ by (Min) and for φ is left-continuous. This proves one implication of (i); the converse of (i) is a consequence of Remark 3.2. \blacksquare

4 Properties of height functions.

In this section we give some simple properties concerning the regularity of height functions in terms of properties of the corresponding trees. We also make the connection with an earlier probabilistic approach by Aldous.

Let (T, d) be a real tree. Recall the definition of the length measure ℓ_T on T from the Introduction section. Observe that ℓ_T only relies on the metric structure. That the tree has finite length should be “read” from any height function coding the tree. More precisely, let $h \in \mathcal{H}_M$. We set for any $0 \leq a \leq b \leq M$

$$v(h, [a, b]) = \sup \sum_{1 \leq i \leq n} |h(t_i) - h(t_{i-1})|,$$

where the supremum is taken over all subdivisions $t_0 = a < t_1 < \dots < t_n = b$. The (possibly infinite) quantity $v(h, [a, b])$ is then the total variation of h over $[a, b]$. Let $r \in [0, M]$ and let $t_0 = 0 < t_1 < \dots < t_n = r$. We denote by $\text{Span}_h(t_1, \dots, t_n)$ the subtree of T_h spanned by the vertices $p_h(t_1), \dots, p_h(t_n)$ and the root ρ_h :

$$\text{Span}_h(t_1, \dots, t_n) = \bigcup_{1 \leq i \leq n} \llbracket \rho_h, p_h(t_i) \rrbracket.$$

Fix $\epsilon > 0$. For any $1 \leq i \leq n-1$, we can find $s_i(\epsilon) \in [t_i, t_{i+1}]$ such that

$$h(s_i(\epsilon)) \leq \frac{\epsilon}{n} + \inf_{s \in [t_i, t_{i+1}]} h(s). \quad (63)$$

Now, think of the rooted ordered subtree $\text{Span}_h(t_1, \dots, t_n)$ as a planar tree, namely a tree embedded in the clockwise oriented half-plane; imagine a particle that continuously moves on it at unit speed, that starts at the root ρ_h and that backtracks as less as possible. The total amount of time needed by the particle to cover the tree and to go back to the root is twice the total length of $\text{Span}_h(t_1, \dots, t_n)$. More precisely the function recording the distance of the particle from the root is the piecewise linear continuous function with slope $+1$ or -1 that goes through the values

$$0, h(t_1), \inf_{s \in [t_1, t_2]} h(s), h(t_2), \dots, \inf_{s \in [t_{n-1}, t_n]} h(s), h(t_n), 0.$$

If we look at the particle until it visits for the last (and perhaps also the first) time $p_h(t_n)$, then all the point of $\text{Span}_h(t_1, \dots, t_n)$ have been visited twice or more except the points of $\llbracket \rho_h, p_h(t_n) \rrbracket \setminus \text{Br}(T)$, the leaves of $\text{Span}_h(t_1, \dots, t_n)$ and possibly the root. Deduce from the previous observations that

$$\begin{aligned} \sum_{1 \leq i \leq n} |h(t_i) - h(t_{i-1})| &\leq d_h(\rho_h, p_h(t_1)) + d_h(p_h(t_1), p_h(t_2)) + \dots + d_h(p_h(t_{n-1}), p_h(t_n)) \\ &\leq 2\ell_T(\text{Span}_h(t_1, \dots, t_n)) - h(r) \end{aligned}$$

since $d_h(\rho_h, p_h(t_n)) = h(r)$. Now deduce from (63) that

$$\begin{aligned} h(t_1) + |h(t_1) - h(s_1(\epsilon))| + |h(t_2) - h(s_1(\epsilon))| + \dots + |h(t_n) - h(s_{n-1}(\epsilon))| &\geq \\ d_h(\rho_h, p_h(t_1)) + d_h(p_h(t_1), p_h(t_2)) + \dots + d_h(p_h(t_{n-1}), p_h(t_n)) &- \epsilon. \end{aligned}$$

Consequently

$$v(h, [0, r]) = 2\ell_T(p_h([0, r])) - h(r)$$

for

$$\sup \ell_T(\text{Span}_h(t_1, \dots, t_n)) = \ell_T(p_h([0, r])),$$

where the supremum is taken over all the subdivisions $t_0 = 0 < t_1 < \dots < t_n = r$. This implies the following proposition.

Proposition 4.1 *Let (T, d) be a compact real tree. $\ell_T(T)$ is a finite quantity iff there exists a height function $h \in \mathcal{H}$ with bounded variation such that (T, d) and (T_h, d_h) are isometric.*

Remark 4.1 It is easy to check that if the length $\ell_T(T)$ is finite then Hausdorff and packing dimensions agree and are equal to 1. \square

We now discuss continuity properties of height processes. Let us first prove the following lemma.

Lemma 4.2 *Let $h \in \mathcal{H}$. We can always find a continuous $c \in \mathcal{H}$ such that $(T_h, d_h, \rho_h, \leq_h)$ and $(T_c, d_c, \rho_c, \leq_c)$ are isometric.*

Proof: Let us first mention that c is in general not unique and that it may never satisfy (Min) (see Comment 1.2). Here we provide one possible function c by interpolating the jumps of h in an order-preserving way. Denote by t_n , $n \geq 1$, a sequence of $[0, \zeta(h)]$ containing all the jump-times of h ; set

$$\psi(t) = t + \sum_{n \geq 1} 2^{-n} \mathbf{1}_{[0, t]}(t_n) \quad \text{and} \quad \Lambda(s) = \inf\{t \in [0, M] : \psi(t) > s\}.$$

Clearly ψ is increasing and right-continuous on $[0, M]$. Thus, Λ is well defined and continuous on $[0, M+1]$. Moreover, $\lim_{s \rightarrow M+1} \Lambda(s) = M$. Set $a = \psi(\Lambda(s)-)$ and $b = \psi(\Lambda(s))$. If $a < b$, then $a \leq s \leq b$ and for any $u \in (a, b)$, we get $\Lambda(u) = \Lambda(s)$. In that case define

$$\theta(u) = (h(\Lambda(s)) - h(\Lambda(s)+)) \frac{u - a}{b - a}.$$

If otherwise $a = b$, then set $\theta(s) = 0$. Then, define $c(s) = h(\Lambda(s)) - \theta(s)$, $s \in [0, M+1]$. Check that c is continuous and that $(T_h, d_h, \rho_h, \leq_h)$ and $(T_c, d_c, \rho_c, \leq_c)$ are isometric. \blacksquare

Let (T, d, ρ, \leq, μ) be a structured tree such that \leq satisfies (Or1) and (Or2) and such that μ satisfies (Mes). Recall that $\phi : [0, \mu(T)] \rightarrow T$ stands for the exploration mapping associated with μ defined in Definition 3.1. Fix $t \in [0, \mu(T)]$. It is easy to check that $\phi(t) \neq \phi(t+)$ iff no subtrees are grafted on the “right side” of the branch $\llbracket \phi(t+), \phi(t) \rrbracket$. Namely, $\phi(t) \neq \phi(t+)$ iff

$$\{\sigma \in T : \phi(t) < \sigma \quad \text{and} \quad \sigma \wedge \phi(t) \in \llbracket \phi(t+), \phi(t) \rrbracket\} = \emptyset.$$

Thus, the height process $h \in \mathcal{H}$ associated with the structured tree (T, d, ρ, \leq, μ) by Theorem 1.1 is continuous iff for any $\sigma_1 \in T$ and for any $\sigma_2 \in \llbracket \rho, \sigma_1 \rrbracket$,

$$\{\sigma \in T : \sigma_1 < \sigma \quad \text{and} \quad \sigma \wedge \sigma_1 \in \llbracket \sigma_2, \sigma_1 \rrbracket\} \neq \emptyset. \quad (64)$$

It implies that the leaves of T are dense:

$$\overline{\text{Lf}(T)} = T. \quad (65)$$

(Indeed let $\gamma \in \llbracket \sigma_2, \sigma_1 \rrbracket$ and fix $\epsilon > 0$; (64) implies that $\text{Br}(T)$ is dense in $\llbracket \sigma_2, \sigma_1 \rrbracket$; since (T, d) is compact, there are only finitely many connected components of $T \setminus \llbracket \sigma_2, \sigma_1 \rrbracket$ with a diameter larger than ϵ ; Thus the set of points in $\llbracket \sigma_2, \sigma_1 \rrbracket$ on which are grafted the connected components of $T \setminus \llbracket \sigma_2, \sigma_1 \rrbracket$ with diameter $\leq \epsilon$ is dense in $\llbracket \sigma_2, \sigma_1 \rrbracket$; consequently we can find a leaf $\sigma \in \text{Lf}(T)$ in such a component such that $d(\sigma, \gamma) \leq 2\epsilon$; it implies that the leaves are dense in the skeleton of T , which proves (65).)

Conversely, we prove the following proposition.

Proposition 4.3 *Let (T, d, ρ) be a compact rooted real tree such that $\overline{\text{Lf}(T)} = T$. Then a.s. for any finite Borel measure μ whose topological support is T , the height process h_{Sh} associated with the structured tree $(T, d, \rho, \leq_{\text{Sh}}, \mu)$ by Theorem 1.1 is continuous.*

Proof: Clearly, $\overline{\text{Lf}(T)} = T$ implies

$$\forall \sigma_1 \in T, \forall \sigma_2 \in \llbracket \rho, \sigma_1 \rrbracket, \exists \sigma \in T \setminus \llbracket \sigma_2, \sigma_1 \rrbracket : \sigma \wedge \sigma_1 \in \llbracket \sigma_2, \sigma_1 \rrbracket.$$

Arguments similar to those used in the proof of Proposition 2.8 imply that a.s. the ordered tree $(T, d, \rho, \leq_{\text{Sh}})$ satisfies (64). The details are left to the reader. \blacksquare

Remark 4.2 Although (65) does not depend on any measure on T , note that if there exists a measure μ on (T, d, ρ) such that

$$\text{supp } \mu = T \quad \text{and} \quad \mu(\text{Sk}(T)) = 0, \quad (66)$$

then (T, d, ρ) satisfies (65). Observe that if in addition μ is non-atomic, then any height function $h \in \mathcal{H}$ coding (T, d, ρ, \leq, μ) fulfils (Min) and is therefore unique. \square

Conversely we have the following proposition.

Proposition 4.4 *Let (T, d, ρ) be a compact rooted real tree such that $\overline{\text{Lf}(T)} = T$. Then there exists a probability measure μ on T that satisfies (CT1), (CT2) and (CT3).*

Proof: We construct such a probability measure thanks to a specific splitting of T that we first explain: Let $Q_\emptyset = (q_n^{(\emptyset)}; n \geq 1)$ be a dense sequence of distinct leaves of T . Let C_k^o , $k \geq 1$ be the connected components of $T \setminus \llbracket \rho, q_1^{(\emptyset)} \rrbracket$ listed in such a way that for any $1 \leq k < l$,

$$\min \{n \geq 1 : q_n^{(\emptyset)} \in C_k^o\} < \min \{n \geq 1 : q_n^{(\emptyset)} \in C_l^o\}.$$

Fix $k \geq 1$. Denote by C_k the closure of C_k^o and denote by σ_k the vertex of $\llbracket \rho, q_1^{(\emptyset)} \rrbracket$ such that $C_k = C_k^o \cup \{\sigma_k\}$. We also define $Q_k = (q_i^{(k)}; i \geq 1)$ by $q_i^{(k)} = q_{n(i)}^{(\emptyset)}$, where $n(i)$, $i \geq 1$, is the increasing sequence of indices $n \geq 1$ such that $q_n^{(\emptyset)} \in C_k^o$. Then, we have defined

$$\text{Split}((T, d, \rho); Q_\emptyset) = ((C_k, d, \sigma_k); Q_k) ; k \geq 1).$$

We recursively define $((C_u, d, \sigma_u); Q_u)$ for any word $u \in \mathbb{U}$ in the following way:

$$\text{Split}((C_u, d, \sigma_u); Q_u) = ((C_{(u,k)}, d, \sigma_{(u,k)}); Q_{(u,k)}) ; k \geq 1),$$

where (u, k) stands for the concatenation of the word u with the single letter word k . Observe that for any $u = (v, w)$ with $v, w \in \mathbb{U}$ we get

$$C_u \subset C_v \quad (67)$$

and

$$\sigma_u \in C_v^o \quad \text{if} \quad w \neq \emptyset, \quad (68)$$

where C_v^o stands for the interior of the compact set C_v . Note that for any $n \geq 1$

$$\bigcup_{\substack{u \in \mathbb{U} \\ |u|=n}} C_u^o = T \setminus \bigcup_{\substack{u \in \mathbb{U} \\ |u| \leq n}} \llbracket \rho, q_1^{(u)} \rrbracket \quad (69)$$

and by (68) we also get

$$\bigcup_{\substack{u \in \mathbb{U} \\ |u|=n}} C_u \subset \bigcup_{\substack{v \in \mathbb{U} \\ |v|=n-1}} C_v^o. \quad (70)$$

Thus

$$\bigcap_{n \geq 1} \bigcup_{\substack{u \in \mathbb{U} \\ |u|=n}} C_u = T \setminus \bigcup_{u \in \mathbb{U}} \llbracket \rho, q_1^{(u)} \rrbracket. \quad (71)$$

Now observe that

$$\{q_1^{(u)} ; u \in \mathbb{U}\} = \{q_n^{(\emptyset)} ; n \geq 1\}. \quad (72)$$

Then we get by (1)

$$\bigcap_{n \geq 1} \bigcup_{\substack{u \in \mathbb{U} \\ |u|=n}} C_u = \text{Lf}(T) \setminus \{q_n^{(\emptyset)} ; n \geq 1\}. \quad (73)$$

Denote by \mathbb{U}_∞ the set of the positive integers valued sequences. Let $v_\infty = (v_\infty(n); n \geq 1)$ be in \mathbb{U}_∞ . Set $u_n = (v_\infty(1), \dots, v_\infty(n))$ and define the non-empty compact C_{v_∞} by

$$C_{v_\infty} = \bigcap_{n \geq 1} C_{u_n} \subset \text{Lf}(T) \setminus \{q_n^{(\emptyset)} ; n \geq 1\}.$$

Suppose that C_{v_∞} contains two distinct leaves σ and σ' . There exist $n, n' \geq 1$ such that

$$d(\sigma, \sigma \wedge q_n^{(\emptyset)}) < \frac{1}{3}d(\sigma, \sigma \wedge \sigma') \quad \text{and} \quad d(\sigma', \sigma' \wedge q_{n'}^{(\emptyset)}) < \frac{1}{3}d(\sigma', \sigma \wedge \sigma').$$

It implies that there exist two distinct words $u, u' \in \mathbb{U}$ such that

$$\sigma \in C_u^o, \sigma' \in C_{u'}^o \quad \text{and} \quad C_u^o \cap C_{u'}^o = \emptyset,$$

which rises a contradiction. Consequently C_{v_∞} reduces to a single point denoted by $\xi(v_\infty)$. Moreover, deduce from (73) that ξ define a bijective map from \mathbb{U}_∞ onto $\text{Lf}(T) \setminus \{q_n^{(\emptyset)} ; n \geq 1\}$. In addition observe that for any $u = (k_1, \dots, k_n) \in \mathbb{U}$ with $k_1, \dots, k_n \geq 1$, we get

$$\xi^{-1} \left(C_u \cap (\text{Lf}(T) \setminus \{q_n^{(\emptyset)} ; n \geq 1\}) \right) = \{v_\infty \in \mathbb{U}_\infty : v_\infty(i) = k_i, 1 \leq i \leq n\}.$$

It implies that ξ is measurable when \mathbb{U}_∞ is equipped with the sigma-field generated by the applications $v_\infty \rightarrow v_\infty(n)$, $n \geq 1$, and when $\text{Lf}(T) \setminus \{q_n^{(\emptyset)} ; n \geq 1\}$ is equipped with the trace of the Borel sigma-field.

Let $\mathbf{p} = (p_i; i \geq 1)$ be a probability distribution on the positive integers such that $p_i > 0$, $i \geq 1$. Let $V = (\kappa_n; n \geq 1)$ be a sequence of i.i.d random variables distributed in accordance with \mathbf{p} . Denote by μ the distribution of $\xi(V)$. Clearly μ satisfies (CT3). Let $\sigma = \xi(v_\infty)$. Observe that

$$\mu(\{\sigma\}) = \mathbb{P}(\xi(V) = \xi(v_\infty)) = \lim_{n \rightarrow \infty} \mathbb{P}(v_\infty(i) = \kappa_i ; 1 \leq i \leq n) = 0.$$

Thus, μ satisfies (CT2).

For any $n \geq 1$ we set $u_n = (v_\infty(1), \dots, v_\infty(n))$, then $\{\sigma\} = \bigcap C_{u_n}$, by definition of $\xi(v_\infty) = \sigma$. It implies that the diameter of C_{u_n} goes to zero. Thus, for any $\epsilon > 0$, there exists $n(\epsilon)$ such that $C_{u_{n(\epsilon)}}$ is contained in the open ball $B(\sigma, \epsilon)$ centered at σ with radius ϵ . Consequently,

$$\mu(B(\sigma, \epsilon)) \geq \mu(C_{u_{n(\epsilon)}}) = p_{v_\infty(1)} p_{v_\infty(2)} \cdots p_{v_\infty(n(\epsilon))} > 0 ,$$

which implies (CT1). This completes the proof of the proposition. \blacksquare

Let us consider a continuum tree (T, d, ρ, μ) . We now make the connection with an earlier work of Aldous (namely Theorem 15 in [5]) that provides a randomized construction of the height function of continuum trees. This construction detailed in the proof of Theorem 15 in [5] can be rephrased as follows:

- Let Σ_n , $n \geq 1$, be an i.i.d. sequence of points in T with distribution μ . Since (T, d, ρ, μ) is a continuum tree, then a.s. the Σ_n 's are distinct leaves and they form a dense subset of T .
- We equip the continuum tree (T, d, ρ, μ) with the random uniform shuffling \leq_{Sh} that is assumed to be independent of the sequence Σ_n , $n \geq 1$.
- We set $\Sigma_0 = \rho$. For any $n \geq 0$, we define a random number U_n in $[0, 1]$ as follows:
 - We set $U_0 = 0$; we also assume that U_1 is independent of the sequence Σ_n , $n \geq 1$, and that U_1 is uniformly distributed in $[0, 1]$.
 - Suppose that U_1, \dots, U_n have been defined; there are two cases. Either there exists a pair $k_1, k_2 \in \{0, \dots, n\}$ such that Σ_{n+1} is the unique point $\sigma \in \{\Sigma_0, \dots, \Sigma_{n+1}\}$ such that $\Sigma_{k_1} <_{\text{Sh}} \sigma <_{\text{Sh}} \Sigma_{k_2}$; in that case, pick U_{n+1} uniformly at random in the closed interval whose ends are U_{k_1} and U_{k_2} . Either

$$\forall k \in \{0, \dots, n\}, \quad \Sigma_k <_{\text{Sh}} \Sigma_{n+1} ;$$

in that case, pick U_{n+1} uniformly at random in the interval $[\max_{0 \leq k \leq n} U_k, 1]$.

Now set for any $t \in [0, 1]$,

$$f(t) = \limsup_{\epsilon \rightarrow 0} \{d(\rho, \Sigma_n), n \geq 0 : U_n \in [t - \epsilon, t + \epsilon]\} .$$

Then Theorem 15 [5] implies that a.s. f is a continuous function such that (T_f, d_f) and (T, d) are isometric and it is clear from Theorem 1.1 that f is the (unique, by Proposition 4.3 and Remark 4.2) height function associated with the structured tree $(T, d, \rho, \leq_{\text{Sh}}, \mu)$.

Consequently, all height functions constructed thanks to Theorem 15 in [5] coincide with the construction given by Theorem 1.1: in particular, it is the case of the normalized Brownian excursion that encodes the Continuum Random Tree; it is also the case of the height functions of the Inhomogeneous Continuum Random Trees given in [2] and of the height functions of the genealogical tree of stable fragmentations in [25].

Lévy trees introduced by Le Gall and Le Jan in [35] generalize the Brownian tree. They are constructed via the coding by the so called *Height Process* that is a local-time functional of

a spectrally positive Lévy process. Lévy trees can be seen as family trees of *continuous states branching processes* that have been introduced by Jirina and Lamperti (see [28, 30] and also [7]). The Lévy trees are the scaling limits of the discrete Galton-Watson trees (see [15, 17, 18] for a detailed account on that topics). When the underlying branching process a.s. dies out in finite time, then the Height Process is continuous with compact support and the Lévy tree coded by this process is a continuum tree $(\mathbf{T}, \mathbf{d}, \rho, \mathbf{m})$. Moreover, given $(\mathbf{T}, \mathbf{d}, \rho, \mathbf{m})$ the order induced by the Height Process corresponds to a uniform random shuffling. Consequently, if we fix the structured tree $(\mathbf{T}, \mathbf{d}, \rho, \leq, \mathbf{m})$ coded by a sample path of the Height Process, then the height function given by Theorem 1.1 coincides with the Height Process itself.

In all these examples of random trees, order does not really matter. Let us end the paper with an example of random tree where the role played by the order is crucial. Let $X = (X(t), t \geq 0)$ be a Lévy process without negative jumps and started at $X(0) = x > 0$. We assume that X does not drift to $+\infty$ so that the stopping time M given by

$$M = \inf\{t \geq 0 : X(t) = 0\}$$

is a.s. finite. Let us set

$$\mathbf{h}(t) = X(M - t), \quad t \in [0, M].$$

Then, $\mathbf{h} \in \mathcal{H}$. We denote by $(\mathbf{T}, \mathbf{d}, \rho, \leq, \mu)$ the random structured tree coded by \mathbf{h} . When X is a compound Poisson process with unit drift, then X can be interpreted as the load of a Last-In-First-Out M/G/1 queueing system and the underlying tree is given by the following rule: we say that Client (a) is the child of Client (b) if Client (b) was currently served when Client (a) arrived in the line (see [35, 37] for more details). The underlying tree can also be seen as the life-time tree of a Crump-Mode-Jagers branching process (see [27] or [16] for a connections with Lévy processes).

Here we consider the case of a Lévy process X for which points are regular and instantaneous, namely a.s.

$$\forall \epsilon > 0, \quad \inf_{0 \leq s \leq \epsilon} X(s) < X(0) < \sup_{0 \leq s \leq \epsilon} X(s).$$

It is equivalent for the Lévy process to have infinite variation paths (we refer to the book of Bertoin [6] Chapter VII Corollary 5 for details). By an easy time-reversal argument, we can show that for any $t_0 > 0$, a.s. we get

$$\forall \epsilon > 0, \quad \inf_{t_0 \leq s \leq t_0 + \epsilon} X(s) < X(t_0) \quad \text{and} \quad \inf_{t_0 - \epsilon \leq s \leq t_0} X(s) < X(t_0).$$

This implies that a.s. μ is a non-atomic measure and that $\mu(\text{Sk}(\mathbf{T})) = 0$. Thus, $(\mathbf{T}, \mathbf{d}, \rho, \mu)$ is a continuum random tree.

Now, fix $(\mathbf{T}, \mathbf{d}, \rho, \leq, \mu)$ and denote by \leq_{sh} a random uniform shuffling of $(\mathbf{T}, \mathbf{d}, \rho, \mu)$. Then, Proposition 4.3 implies that the new height function \mathbf{h}_{sh} associated by Theorem 1.1 with $(\mathbf{T}, \mathbf{d}, \rho, \leq_{\text{sh}}, \mu)$ is continuous. Thus, it is a continuous rearrangement of the Lévy process $X = (X(t); 0 \leq t \leq M)$ coding the same measured compact rooted real tree. Excepted in the Brownian case, the distribution of \mathbf{h}_{sh} does not seem to be simple to characterize.

References

- [1] ALDOUS, D., MIERMONT, G., AND PITMAN, J. Brownian bridge asymptotics for random p-mappings. *Electronic J. Probab.* 9 (2004), 37–56.
- [2] ALDOUS, D., MIERMONT, G., AND PITMAN, J. The exploration process of inhomogeneous continuum random trees and an extension of Jeulin’s local time identity. *Probab. Th. Related Fields* 129 (2004), 182–218.
- [3] ALDOUS, D., MIERMONT, G., AND PITMAN, J. Weak convergence of random p-mappings and the exploration process of inhomogeneous continuum random trees. *Probab. Th. Related Fields* 133 (2005), 1–17.
- [4] ALDOUS, D. J. The continuum random tree I. *Ann. Probab.* 19 (1991), 1–28.
- [5] ALDOUS, D. J. The continuum random tree III. *Ann. Probab.* 21 (1993), 248–289.
- [6] BERTOIN, J. *Lévy Processes*. Cambridge Univ. Press, 1996.
- [7] BINGHAM, N. H. Continuous branching processes and spectral positivity. *Stochastic Process. Appl.* 4 (1976), 217–242.
- [8] BUNEMAN, P. A note on the metric properties of trees. *J. Combinatorial Theory Ser. B* 17 (1974), 48–50.
- [9] BURAGO, D. BURAGO, Y., AND IVANOV, S. *A Course in Metric Geometry*, vol. 33. AMS, Boston, 2001.
- [10] CHISWELL, I. *Introduction to Λ -trees*. World Scientific Publishing Co., Inc, River Edge, 2001.
- [11] CROYDON, D. Measure and heat kernel estimates for the continuum random tree. *preprint* (2005).
- [12] DRESS, A. Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: A note on combinatorial properties of metric spaces. *Adv. Math.* 53 (1984), 321–402.
- [13] DRESS, A., MOULTON, V., AND TERHALLE, W. T-theory: an overview. *European J. Combin.* 17 (1996), 161–175.
- [14] DRESS, A., AND TERHALLE, W. The real tree. *Adv. Math.* 120 (1996), 283–301.
- [15] DUQUESNE, T. A limit theorem for the contour process of conditioned Galton-Watson trees. *Ann. Probab.* 31, 2 (2003), 996–1027.
- [16] DUQUESNE, T., AND LAMBERT, A. Work in progress. - (2005).
- [17] DUQUESNE, T., AND LE GALL, J.-F. *Random Trees, Lévy Processes and Spatial Branching Processes*. Astérisque no 281, 2002.

- [18] DUQUESNE, T., AND LE GALL, J.-F. Probabilistic and fractal aspects of Lévy trees. *To appear in Probab. Theorey and Rel. Fields* (2004).
- [19] DUQUESNE, T., AND LE GALL, J.-F. The Hausdorff measure of stable trees. *preprint* (2005).
- [20] DUQUESNE, T., AND WINKEL, M. Growth of Lévy trees. *preprint* (2005).
- [21] EVANS, S. Snakes and spiders: Brownian motion on real trees. *Probab. Theory Related Fields* 117, 3 (2000), 361–386.
- [22] EVANS, S., PITMAN, J., AND WINTER, A. Rayleigh processes, real trees, and root growth with re-grafting. *To appear in Probab. Th. Rel. Fields* (2005).
- [23] EVANS, S., AND WINTER, A. Subtree prune and re-graft: a reversible real tree valued Markov process. *preprint* (2005).
- [24] FELSENSTEIN, J. *Inferring Phylogenies*. Sinauer Associates, Sunderland, Massachusetts, 2003.
- [25] HAAS, B., AND MIERMONT, G. The genealogy of self-similar fragmentations with negative index as a continuum random tree. *Electr. J. Probab.* 9 (2004), 57–97.
- [26] HAMBLY, B., AND LYONS, T. Uniqueness for the signature of a path of bounded variation and continuous analogues for the free group. *Preprint* (2004).
- [27] JAGERS, P. General branching processes as Markov fields. *Stoch. Proc. Appl.* 32 (1989), 213–224.
- [28] JIRINA, M. Stochastic branching processes with continous state-space. *Czech. Math. J.* 8 (1958), 292–313.
- [29] KREBS, W. Brownian motion on the continuum tree. *Probab. Theory Rel. Fields* 101, 3 (1995), 421–433.
- [30] LAMPERTI, J. The limit of a sequence of branching processes. *Z. Wahrsch. Verw. Gebiete* 7 (1967), 271–288.
- [31] LE GALL, J.-F. 2005. Manuscript notes.
- [32] LE GALL, J.-F. Brownian excursions, trees and measure-valued branching processes. *Ann. Probab.* 19 (1991), 1399–1439.
- [33] LE GALL, J.-F. A class of path-valued Markov processes and its applications to superprocesses. *Prob. Th. Rel. Fields* 95 (1993), 25–46.
- [34] LE GALL, J.-F. The uniform random tree in a Brownian excursion. *Probab. Theory and Related Fields* 96 (1993), 369–383.

- [35] LE GALL, J.-F., AND LE JAN, Y. Branching processes in Lévy processes: the exploration process. *Ann. Probab.* 26-1 (1998), 213–252.
- [36] LE GALL, J.-F., AND LE JAN, Y. Branching processes in Lévy processes: Laplace functionals of snakes and superprocesses. *Ann. Probab.* 26 (1999), 1407–1432.
- [37] LIMIC, V. A LIFO queue in heavy traffic. *Ann. Appl. Probab.* 11 (2001), 301–331.
- [38] MAYER, J., AND OVERSTEEGEN, L. A topological characterization of \mathbb{R} -trees. *Trans. Amer. Math. Soc.* 320 (1990), 395–415.
- [39] MIERMONT, G. Self-similar fragmentations derived from the stable tree I: splitting at heights. *Probab. Theory Relat. Fields* 127, 3 (2003), 423–454.
- [40] MIERMONT, G. Self-similar fragmentations derived from the stable tree II: splitting at nodes. *Probab. Theory Relat. Fields* 131, 3 (2005), 341–375.
- [41] NEVEU, J. Arbres et processus de Galton-Watson. *Ann. Inst. H. Poincaré* 26 (1986), 199–207.
- [42] SEMPLE, C., AND STEEL, M. *Phylogenetics*, vol. 24 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2003.
- [43] SIMÕES PEREIRA, J. M. S. A note on the tree realizability of a distance matrix. *J. Combinatorial Theory* 6 (1969), 303–310.
- [44] WEILL, M. Regenerative real trees. *preprint* (2005).
- [45] ZARECKII, K. A. Constructing a tree on the basis of a set of distances between the hanging vertices. *Uspehi Mat. Nauk* 20, 6 (1965), 90–92.